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A REPRESENTATION FOR BOOLEAN ALGEBRAS.*

By FRANKLIN HAIMO.

1. The content. The success of Stone [5] in representing Boolean algebras by fields of sets leads one to search for other representations. (See also [2, p. 159], [4].) Using splitting endomorphisms on a type of Abelian group with an order relation, we are led to a faithful representation of a given Boolean algebra $\mathfrak{B}$. The group employed will be called a Boolean group. A special case of a Boolean group is a Boolean ring, and we shall show that $\mathfrak{B}$ can be represented by a set of endomorphisms on its corresponding Boolean ring. If $\mathfrak{B}$ is complete the group on which it operates will be lower complete and upper conditionally complete. The Boolean groups are themselves special cases of a wider class of groups with an order relation, the vector ordered groups. Vector ordered groups generalize direct sums of Abelian groups, and their splitting endomorphisms form Boolean algebras.

A simple example illustrates virtually the entire theory: Let $\mathfrak{J}_1$ be the ring of integers modulo 2, and let $\mathfrak{G}$ be the Cartesian product of some infinite collection of copies of $\mathfrak{J}_2$. $\mathfrak{G}$ can be made into the strong direct sum of the $\mathfrak{J}_2$ by introducing component-wise addition and multiplication. Then $\mathfrak{G}$ becomes a Boolean ring with a unity. It is also a Boolean algebra [2, p. 154] if one writes $x \geq y$ whenever each component of $y \in \mathfrak{G}$ equals the corresponding component of $x \in \mathfrak{G}$ or is zero. Let $\mathfrak{P}(\mathfrak{G})$ be the set of all endomorphisms V of the group structure of $\mathfrak{G}$ for which $V^2 = V$ and $V(x) \leq x$ for every $x \in \mathfrak{G}$. If an order relation is defined in $\mathfrak{P}$ in the obvious way, $\mathfrak{P}$ becomes a Boolean algebra isomorphic to the Boolean algebra $\mathfrak{G}$, so that the algebra $\mathfrak{G}$ is represented by a set of operators on the group $\mathfrak{G}$. If we replace each summand $\mathfrak{J}_2$ by $\mathfrak{J}$, the group of integers, the corresponding strong direct sum group $\mathfrak{H}$ can be ordered component-wise as was done for $\mathfrak{G}$. Then $\mathfrak{P}(\mathfrak{H}) \cong \mathfrak{P}(\mathfrak{G})$. In fact, we can reconstruct both $\mathfrak{H}$ and $\mathfrak{G}$ from $\mathfrak{P}$ by assigning coefficients from $\mathfrak{J}$ (respectively, from $\mathfrak{J}_2$) to the atoms of $\mathfrak{P}$. The groups $\mathfrak{G}$ and $\mathfrak{H}$ are examples of Boolean groups, to be defined below. If the members of a proper subcollection of the summands $\mathfrak{J}_2$ are replaced by $\mathfrak{J}$'s and if the remaining summands are unaltered, the resulting strong direct sum group $\mathfrak{K}$, ordered as above, has the property $\mathfrak{P}(\mathfrak{K}) \cong \mathfrak{P}(\mathfrak{G})$. $\mathfrak{K}$ is an example of a vector ordered group of a type which we shall designate as "strong." The

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use of the atoms of $\mathcal{P}$ is inessential, as we shall see below. For instance, had we reduced the group $\mathfrak{G}$ modulo the subgroup $\mathfrak{G}^w$, the weak direct sum of the same summands, we would have been led to an atomless example of the same theory. Part I of this paper will be concerned with the (strong) vector ordered groups. In Part II we shall prove the representation theorems, using results from Part I when needed.

2. Notation. Groups, subgroups and "complexes" will be denoted by Gothic capitals; their elements, however, will be given by small Latin letters, their endomorphisms by Latin capitals. Although early Greek small letters are used in Part I for indexing, in Part II they usually denote elements of Boolean algebras. These latter algebras and some other partly ordered sets will often be given by Latin script capitals. Exceptions to the above conventions and additional notation will be clear from the context. The basic terminology is essentially that of G. Birkhoff [2].

Part I.

3. The concepts $\perp$ and $\|$. Let $\mathcal{S}$ be a partly ordered set with universal infimum 0. If $x, y \in \mathcal{S}$, we say that $x \perp y$ (x is perpendicular to y) if (1) $z \in \mathcal{S}$, $z \leq x, z \leq y$ imply $z = 0$, and (2) there exists $w \in \mathcal{S}$ such that $x \leq w, y \leq w$. It is clear that $\perp$ is a symmetric relation on $\mathcal{S}$ and that $0 \perp x$ for every $x \in \mathcal{S}$. $x \perp x$ if, and only if, $x = 0$. For subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$, we say that $\mathcal{A} \perp \mathcal{B}$ if $a \in \mathcal{A}, b \in \mathcal{B}$ imply $a \perp b$.

We say that $x \parallel y$ (x is parallel to y) if (1) $z \leq x$, $z \perp y$ imply $z = 0$, and (2) $w \perp x$, $w \leq y$ imply $w = 0$. $\parallel$ is a reflexive and symmetric relation on $\mathcal{S}$. $\mathcal{A} \parallel \mathcal{B}$ is defined in the obvious way.

4. Vector ordered groups. Let $\mathfrak{G}$ be an additive Abelian group, partly ordered by a relation $\leq$, (where $x \leq y$ may be written $y \geq x$), subject to the following axioms:

(1) $x \geq 0$, for every $x \in \mathfrak{G}$ (where 0 is the additive identity of $\mathfrak{G}$).

(2) $x \geq y$ implies $y \perp (x - y)$.

(3) $x \perp y$ implies $x + y \geq x, y$.

(4) $x + y \geq x, y$ imply that $x + y = x \cup y$, the least upper bound (l. u. b.) of x and y.

(5) For fixed $a \in \mathfrak{G}$ the sets $\mathfrak{R}_a = [x \mid x \in \mathfrak{G}; x \perp a]$ and $\mathfrak{H}_a = [x \mid x \in \mathfrak{G}; z \leq x, z \perp a$ imply $z = 0]$ are subgroups of $\mathfrak{G}$.

(6) (a) To each $x \in \mathfrak{G}$ there exists at least one maximal element m of $\mathfrak{G}$ with $x \leq m$. (b) Given $a, b \in \mathfrak{G}$, there exists $c \in \mathfrak{G}$ such that $c \leq a$,

$c \perp b$, and such that $d \leq a$, $d \perp b$ imply $d \leq c$. We write $c = a_{\perp b}$. (c) Given $a, b \in \mathfrak{G}$. there exists $c \in \mathfrak{G}$ such that $c \leq a$, $c \leq b$, and such that $d \leq a$, $d \leq b$ imply $d \leq c$. We write $c = ab$.

Abelian groups satisfying (1) through (6) are called vector ordered groups (v. o. groups). For instance, let $\mathfrak{G}$ be the direct sum of a finite number of additive Abelian groups $\mathfrak{G}_i$, $i = 1, 2, \cdots, n$. If $a, b \in \mathfrak{G}$, we say that $a \geq b$ if b can be obtained from a by replacing some (or no) non-zero components of a by zeros, leaving the other components fixed. It is clear that this ordering make $\mathfrak{G}$ a v. o. group.

Axiom (6) (c) will seldom be used and is given mainly for completeness. We shall often replace (6) by:

(6′) Every chain in $\mathfrak{G}$ has a l. u. b. in $\mathfrak{G}$.

Abelian groups satisfying (1) through (5) and (6′) are called strong v. o. groups. We shall show that (6′) and the other axioms imply (6) so that a strong v. o. group is a v. o. group. If one orders a strong direct sum $\mathfrak{G}$ of a set of additive Abelian groups $\mathfrak{G}_a$ in the same fashion as direct sums were ordered above, then $\mathfrak{G}$ becomes a strong v. o. group. Proofs will be carried through for v. o. groups rather than for strong v. o. groups except where the contrary is indicated.

5. Elementary results. The following have completely trivial proofs. Often these results will be used without explicit reference. Those, the proofs of which employ (6), are starred. (a) $x + y \geq x$ is equivalent to $x + y \geq y$. (b) A non-zero, non-maximal element of $\mathfrak{G}$ is $\perp$ to some other non-zero, non-maximal element. (c) The following are equivalent: $x \geq y$, $-x \geq -y$ and $x \geq x - y$. (d) $x + y \geq x$ implies $x \perp y$. (e) $x \perp y$, $y \geq z$ imply $x \perp z$. (f) $x + y \geq z$, $x \perp z$ imply $y \geq z$. (g) $u \leq x + y$, $u \leq y$ and $v \leq x$ imply $u \perp v$. (h) $x \perp y, x, y \leq w$ imply $x + y \leq w$. (i) $w \geq x + y$, $x \perp y$ imply $w \geq x, y$. (j) $x \perp y, x \geq z$ imply $x + y \geq z + y$. (k) $x \perp y$, $x \geq z$, $y \geq w$ imply $x + y \geq z + w$. (l) $x \geq y \geq z$ imply $x - z \geq y - z$. (m) The principal of limited homogeneity: if $a \leq b$, then $a + x \leq b + x$ if, and only if, $x \perp (b - a)$. (n) For every integer n, $x \geq y$ implies $nx \geq ny$. (n′) Let $\mathfrak{F}$ be the subgroup of those elements of $\mathfrak{G}$ which are zero or have finite order, the torsion subgroup of $\mathfrak{G}$. Then $\mathfrak{F}$ is convex in $\mathfrak{G}$. That is, $x \in \mathfrak{G}$, $f \in \mathfrak{F}$, $x \leq f$ imply $x \in \mathfrak{F}$. (o) For given $a \in \mathfrak{G}$, $\mathfrak{R}_a$ and $\mathfrak{H}_a$ are convex, and $\mathfrak{R}_a \cap \mathfrak{H}_a = (0)$, the subgroup of $\mathfrak{G}$ consisting of the element 0 alone. (p) $[x \mid x \in \mathfrak{G}; x \geq a] = a + \mathfrak{R}_a$, the addition being that of complexes. (q) If $x, y \leq z$, there exist $y', x' \in \mathfrak{G}$ such that $y' \perp x$, $x' \perp y$ and $y - x = y' - x'$. (r) For $a \neq 0$, $\mathfrak{R}_a$ contains no maximal element of $\mathfrak{G}$. (s*) If a subset of $\mathfrak{R}_a$

has a l. u. b. in $\mathfrak{G}$, then this l. u. b. is in $\mathfrak{R}_a$. (t) $c \geq a, b$ and v maximal with respect to the property $v \leq a, b$ imply $a - v \perp b$ and $b - v \perp a$. (t') If v is maximal with respect to the property $v \leq a, b$, then $(a - v) \perp b$ if, and only if, there exists $c \in \mathfrak{G}$, $c \geq a - v$, $c \geq b$. (u*) to each non-zero, non-maximal element $x \in \mathfrak{G}$ there exists at least one $x' \in \mathfrak{G}$, non-zero and non-maximal with $x' \perp x$ and x' maximal with respect to this property. (Taking an m from (6)(a), let $x' = m - x$.) (u'*) In (u*), $w \perp x', w \neq x$ imply $w \not\geq x$. (u''*) If n is an integer, $nx \neq x$ implies $nx \not\geq x$.

We can define a new binary operation $+'$ on some of the pairs of elements of $\mathfrak{G}$ as follows: if $x \perp y$, let $x +' y = x + y$. $+'$ is an associative operation where it is applicable. Under $+'$, $\mathfrak{G}$ becomes an ordered (cf. (j), (m) above) groupoid, similar, in some respects, to the groupoid of Brandt [3]. The properties of this groupoid will be discussed elsewhere.

We shall use the axiom of choice in the form of Zorn's Lemma [2, p. 42] whenever it is needed.

LEMMA 1. *The first five axioms and* (6′) *imply* (6) *and a stronger form of* (6)(c):

(6)(c′) *Strong vector ordered* $\mathfrak{G}$ *is lower complete.*

Proof. (6)(a) is immediate from Zorn's Lemma. As for (6)(b), suppose that $a, b \in \mathfrak{G}$. By Zorn's Lemma and (6′), there exists $s \in \mathfrak{G}$, $s \leq a$, $s \perp b$, and s is maximal (see (s*) above) with respect to this property. If $d \leq a$, $d \perp b$, Zorn's Lemma supplies $w \leq d$, s with w maximal with respect to this property. By (t) above, $s \perp d - w$, and $s + d - w \geq s$. Since $s, d, w \in \mathfrak{R}_b$, $s - d + w \perp b$. $w \leq s \leq a$ imply $a \geq s - w$. But $a \geq d$, and, by (t), $d \perp s - w$. Hence $a \geq s + d - w$. By the maximal character of s, $s = s + d - w$, and $w = d$, $d \leq s$, proving the maximum property of s.

To prove (6)(c′), let $\mathfrak{C}$ be a non-void class of elements of $\mathfrak{G}$. The class $\mathfrak{L}$ of lower bounds for all the elements of $\mathfrak{C}$ is non-void sinve $0 \in \mathfrak{L}$. For a given chain $\mathfrak{D} \subset \mathfrak{L}$, (6′) provides a l. u. b. which is seen to be in $\mathfrak{L}$. By Zorn's Lemma, to each $j \in \mathfrak{L}$, there exists $c(j) \in \mathfrak{L}$, where $c(j) \geq j$, and where $c(j)$ is maximal in $\mathfrak{L}$. If c and d are two such maximal elements of $\mathfrak{L}$, then it is easy to show that $c \parallel d$. By an argument similar to the one on $\mathfrak{C}$ we can construct an element v where $v \leq c, d$ and is maximal with respect to this property. c and d have at least one common upper bound. Applying (t) above, $c - v \perp d$. But $c \geq v$ implies $c \geq c - v$. Since $c \parallel d$, $c - v = 0$ and $c = v$. Hence $c \leq d$. Similarly, $d \leq c$ and $c = d$. This establishes c as the g. l. b. of the elements of $\mathfrak{C}$, and $\mathfrak{G}$ is thus lower complete.

6. Endomorphisms and projections. Let $\mathfrak{G}$ be a v. o. group. Consider the following five properties of endomorphisms V. (α) $V(x) \leq x$ for every $x \in \mathfrak{G}$. (β) $V^2 = V$ (V is splitting on $\mathfrak{G}$). (γ) Let the kernel of V be denoted by $\mathfrak{K}(V)$ and set $\mathfrak{H}(V) = [x \mid x \in \mathfrak{G}; V(x) = x]$. Then the subgroups $\mathfrak{H}(V)$ and $\mathfrak{K}(V)$ are elementwise $\perp$. (δ) $x \geq y$ implies $V(x) \geq V(y)$. (ϵ) $x \perp y$ implies $V(x) \perp V(y)$. It is immediate that (δ) and (ϵ) are equivalent, that (α) implies (γ) and (δ) and that (β) and (γ) as a pair imply (α). Likewise, (α) implies (β); for, $V(x) \geq V^2(x)$ so that $V(x) \geq V(x) - V^2(x)$, and $x \geq V(x)$ implies $V(x) \perp x - V(x)$. Therefore, $V(x) - V^2(x) \perp x - V(x)$. But $V(x) - V^2(x) \leq x - V(x)$. This shows that $V(x) - V^2(x) = 0$ for every $x \in \mathfrak{G}$.

An endomorphism V of $\mathfrak{G}$ with property (α) is called a projection,[1] and every projection is a splitting endomorphism. Also $\mathfrak{G} = \mathfrak{H}(V) \oplus \mathfrak{K}(V)$, a direct sum of $\perp$ summands. Let b be a fixed element of $\mathfrak{G}$. Define a a mapping P_b on $\mathfrak{G}$ into $\mathfrak{G}$ by $P_b(x) = x - x_{\perp b}$. For brevity, we set $x_b = x - x^{\perp_b}$. $x_b \leq x$. $x_b = 0$ if, and only if, $x = x_{\perp b}$, that is, if, and only if, $x \perp b$. Hence $\mathfrak{K}(P_b) = \mathfrak{K}_b$. $x_b \perp x_{\perp b}$ and $x_b \in \mathfrak{H}_b$. Thus, $(x_b)_{\perp b} = 0$, and $P^2{}_b = P_b$. Also $P_b(x) = x$ if, and only if, $x_{\perp b} = 0$. This shows that $\mathfrak{H}(P_b) = \mathfrak{H}_b$. Now $x = x_b + x_{\perp b}$. Hence $\mathfrak{G} = \mathfrak{H}_b \oplus \mathfrak{K}_b$. Thus, P_b is an endomorphism and therefore a projection. It is onto $\mathfrak{H}_b$, its invariant set. There is a companion projection P'_b defined by $P'_b(x) = x_{\perp b}$. $\mathfrak{H}(P'_b) = \mathfrak{K}_b$ and $\mathfrak{K}(P'_b) = \mathfrak{H}_b$.

The following eight conditions are equivalent. The proofs are elementary. (1) $a \parallel b$. (2) $\mathfrak{H}_a = \mathfrak{H}_b$. (3) $\mathfrak{K}_a = \mathfrak{K}_b$. (4) a is maximal in $\mathfrak{H}_b$. (5) b is maximal in $\mathfrak{H}_a$. (6) $P_a = P_b$. (7) $a \in \mathfrak{H}_b$ and $b \in \mathfrak{H}_a$. (8) $a - b \in \mathfrak{H}_a \cap \mathfrak{H}_b$.

The convexity of $\mathfrak{H}_b$ shows that $x \in \mathfrak{H}_b$ if, and only if, there exists $x^* \in \mathfrak{G}$ with $x^* \geq x$, $x^* \parallel b$. In fact, for $x \in \mathfrak{H}_b$, $x + P'_x(b)$ is an appropriate x^*. x_b is maximum with respect to being in $\mathfrak{H}_b \cap [y \mid y \in \mathfrak{G}; y \leq x]$. Note that a is maximal in $\mathfrak{H}_a$, and that $\mathfrak{H}_a$ is the cross-cut of all $\mathfrak{K}_x$ where $x \perp a$. $x \perp a$, $a \parallel b$ imply $x \perp b$. It follows immediately that $\parallel$ is an equivalence relation $\mathfrak{G}$. Under $\parallel$, all maximal elements of $\mathfrak{G}$ form an equivalence class. The following are also elementary: (a) b_a, $a_b \in \mathfrak{H}_a \cap \mathfrak{H}_b$. (b) $a_b \parallel b_a$. (c) $a + b - b_a \parallel a + b - a_b$, $(a + P'_a(b) \parallel b + P'_b(a))$. (d) $a + b \geq a_b + b_a$, $a_{\perp b} + b_{\perp a}$, and the latter two sums are $\perp$. In fact, $\mathfrak{H}_a \cap \mathfrak{H}_b \perp \mathfrak{K}_a + \mathfrak{K}_b$ (in the sense of addition of complexes), and the $+$ and $\cap$ can be exchanged in this statement. (e) $a \parallel b$, $a \leq a'$, $b \leq b'$ imply $a + b \leq a' + b'$. (f) If $\mathfrak{G}$ is free of elements of order 2, then $c \geq a, b, a + b$ imply $a \perp b$. (g) $a_b = b_a$ if, and only if, there exists

[1] For similar notions in Hilbert space, see Béla v. Sz. Nagy, "Spektraldarstellung linearer Transformationen," Berlin, 1942.

$c \geq a, b$. Thus $c \geq a, b$ and $a \parallel b$ imply $a = b$. (h) If a subset of $\mathfrak{H}_a$ has a l. u. b. in $\mathfrak{G}$, then this l. u. b. lies in $\mathfrak{H}_a$. (*Cf.* section 5, (*s**).) (i) $a \parallel (\perp) b$ if, and only if $a_c \parallel (\perp) b_c$ for every $c \in \mathfrak{G}$. (j) $a + b \geq c$ implies $c = a_c + b_c$. (k) If y is maximal in $\mathfrak{G}$, then $y_x \parallel x$ for every $x \in \mathfrak{G}$.

THEOREM 1. *Let P be a projection of a* v. o. *group $\mathfrak{G}$. Then $P = P_a$ if, and only if, a is maximal in $\mathfrak{H}(P)$; and for each projection P, there exists $a \in \mathfrak{H}(P)$ such that $P = P_a$.*

Proof. Suppose that a is maximal in $\mathfrak{H}(P)$, that $y \perp a$ and that h is the component of y in $\mathfrak{H}(P)$. Then $h \perp a$, and $h + a \geq a$. But $h + a \in \mathfrak{H}(P)$ so that $h = 0$ by the maximal character of a. Hence, $y \in \mathfrak{R}(P)$, and $\mathfrak{R}_a \subset \mathfrak{R}(P)$. $\mathfrak{R}(P) \perp \mathfrak{H}(P)$, and $a \in \mathfrak{H}(P)$ imply $\mathfrak{R}(P) \subset \mathfrak{R}_a$, so that $\mathfrak{R}_a = \mathfrak{R}(P)$. For $x \in \mathfrak{G}$, $x = P(x) + P'(x)$, where $P'(x) \in \mathfrak{R}(P) = \mathfrak{R}_a$. Therefore, $x_a = P_a(x) = P_a P(x)$, and $x_a \leq P(x)$. But $PP_a(x) = P(x_a) = P(x) - P(x_{\perp a}) = P(x)$, since $x_{\perp a} \in \mathfrak{R}_a = \mathfrak{R}(P)$. Thus $P(x) \leq P_a(x) = x_a$; and $P(x) = P_a(x)$ for every $x \in \mathfrak{G}$. Thus $P = P_a$ if a is maximal in $\mathfrak{H}(P)$. If $P = P_a$, then $\mathfrak{H}(P) = \mathfrak{H}_a$. Since a is maximal in $\mathfrak{H}_a$, a is maximal in $\mathfrak{H}(P)$.

Let b be maximal in $\mathfrak{G}$ and let a be its component in $\mathfrak{H}(P)$. If a is not maximal in $\mathfrak{H}(P)$, then there exist $c \in \mathfrak{H}(P)$, $c \neq a$, $c \geq a$. $b - a \in \mathfrak{R}(P)$, and $\mathfrak{R}(P) \perp \mathfrak{H}(P)$. Hence $c - a \perp b - a$. Also $c - a \perp a$. Thus $c - a \perp (b - a) + a$, or $c - a \perp b$; and $c - a + b \geq b$. Since $c - a \neq 0$, this contradicts the maximal character of b. This shows that a is maximal in $\mathfrak{H}(P)$ so that $P = P_a$, completing the proof.

Now the projections have been put into one-to-one correspondence with the equivalence classes of $\mathfrak{G}$ under $\parallel$, we shall have little difficulty in exhibiting the structure of the set $\mathfrak{P}(\mathfrak{G})$ of projections. We can say that $P_a \leq P_b$ if $P_a(x) \leq P_b(x)$ for every $x \in \mathfrak{G}$. This partly orders $\mathfrak{P}(\mathfrak{G})$. For instance, if $a \geq b$, then $P_a \geq P_b$. More generally, if S and T are endomorphisms of $\mathfrak{G}$, we can say that $S \leq T$ if $S(x) \leq T(x)$ for every $x \in \mathfrak{G}$. P_0 is a universal infimum, which we can denote by 0 without confusion; and $I = P_x$, x maximal in $\mathfrak{G}$, is the identity mapping on $\mathfrak{G}$ and is the universal supremum of $\mathfrak{P}(\mathfrak{G})$. $0 = P_0$ carries all elements of $\mathfrak{G}$ onto 0. In the ordered system $\mathfrak{P}(\mathfrak{G})$ the concepts $\perp$ and $\parallel$ are meaningful and will be used. Products of projections are again projections and are written to the left. That is, $P_a P_b(x) = P_a[P_b(x)]$. Though $P_a + P_b$ is an endomorphism, it need not be a projection. $(x_{\perp a})_{\perp b} \leq x_{\perp b}$. Applying P'_a to both sides, $(x_{\perp a})_{\perp b} \leq (x_{\perp b})_{\perp a}$. This leads to $P'_a P'_b = P'_b P'_a$. If P_r is a projection then there exists $t \in \mathfrak{G}$ such that $P_r = P'_t$. Hence $P_r P_s = P_s P_r$.

The following are equivalent: (1)′ $P_a \perp P_b$ in $\mathfrak{P}(\mathfrak{G})$. (2)′ $a \perp b$.

(3)′ $P_a(x) \perp P_b(x)$ for every $x \in \mathfrak{G}$. (4)′ $\mathfrak{H}_a \cap \mathfrak{H}_b = (0)$. (5)′ $P_a P_b = 0$. From any one of these it follows that $P_{a+b} = P_a + P_b = P_a \cup P_b$, the l. u. b. of P_a and P_b. If $\mathfrak{G}$ has no elements of order 2, then the following can be added to the equivalence list: (6)′ $P_a + P_b$ is a splitting endomorphism of $\mathfrak{G}$. (7)′ $P_a + P_b \leq I$.

We give another set of equivalent conditions: (1)″ $P_a \geq P_b$. (2)″ $P_b = P_a P_b$. (3)″ $b \in \mathfrak{H}_a$. (4)″ $\mathfrak{H}_b \subset \mathfrak{H}_a$. (5)″ $\mathfrak{K}_b \supset \mathfrak{K}_a$. (6)″ There exists $c \in \mathfrak{G}$, $c \parallel b$, $c \leq a$. (7)″ There exists $d \in \mathfrak{G}$, $d \parallel a$, $d \geq b$.

Immediate are (l) $P_a \leq P_b$ and $c_a \leq d \leq c_b$ imply $d_a = c_a$. (m) If $\mathfrak{H}$ and $\mathfrak{K}$ are subgroups, if $\mathfrak{H} \perp \mathfrak{K}$ and if $\mathfrak{G} = \mathfrak{H} \oplus \mathfrak{K}$, a direct sum, then there exists a projection P_a of $\mathfrak{G}$ such that $\mathfrak{H}_a = \mathfrak{H}$ and $\mathfrak{K}_a = \mathfrak{K}$.

THEOREM 2. *Let $\mathfrak{G}$ be a* v. o. *group. Then the $\mathfrak{H}_a$ and $\mathfrak{K}_a$ are* v. o. *groups, strong if $\mathfrak{G}$ is strong, under the order relation hereditary from $\mathfrak{G}$.*

Proof. Verifying the axioms one by one, we see that axiom (1) is trivial. (2) For $x, y \in \mathfrak{H}_a$, $x \geq y$, we have $y \perp x - y$ in $\mathfrak{G}$. Since $x \geq y$, $x - y$, with $x \in \mathfrak{H}_a$, $x \perp x - y$ in $\mathfrak{H}_a$. (3) If $x \perp y$ in $\mathfrak{H}_a$, the convexity of $\mathfrak{H}_a$ shows that $x \perp y$ in $\mathfrak{G}$ so that $x + y \geq x, y$ in $\mathfrak{G}$, and the latter also hold in $\mathfrak{H}_a$. (4) If $x + y \geq x, y$ in $\mathfrak{H}_a$, then the same relations hold in $\mathfrak{G}$ so that $x + y = x \cup y$ in $\mathfrak{G}$. Hence $x + y = x \cup y$ in $\mathfrak{H}_a$. (5) For $c \in \mathfrak{H}_a$ let $\mathfrak{D} = [x \mid x \in \mathfrak{H}_a; x \perp c$ in $\mathfrak{H}_a]$. Since the relation $\perp$ in $\mathfrak{H}_a$ implies the relation $\perp$ in $\mathfrak{G}$, axiom (5) shows that $\mathfrak{D}$ is a subgroup of $\mathfrak{H}_a$. Let $\mathfrak{E} = [x \mid x \in \mathfrak{H}_a; z \leq x, z \perp c$ in $\mathfrak{H}_a$ imply $z = 0]$. It turns out that $\mathfrak{E} = \mathfrak{H}_a \cap \mathfrak{H}_c$, so that $\mathfrak{E}$ is a subgroup of $\mathfrak{H}_a$. (6)(a) For $x \in \mathfrak{H}_a$, there exists $y \in \mathfrak{G}$, y maximal, $y \geq x$. $P_a(y) \geq x$, and $P_a(y)$ is maximal in $\mathfrak{H}_a$. (6)(b) and (c). If $b, c \in \mathfrak{H}_a$, the elements $b_{\perp c}$ and bc are in $\mathfrak{H}_a$. (6′) If $\mathfrak{G}$ is strong, so is $\mathfrak{H}_a$, by (h) above. The same statements are true for $\mathfrak{K}_a$, since every $\mathfrak{K}_a$ is an $\mathfrak{H}_b$, by (m) above.

7. $\mathfrak{P}(\mathfrak{G})$ as a Boolean algebra. Let us order the projections as above. Since $b_a \perp b_{\perp a}$, $P_b = P_c + P_{c'}$ where $c = b_a$, $c' = b_{\perp a}$. $P_a P_b = P_a P_c + P_a P_{c'} = P_a P_c + 0 = P_c$, by (2)″ and (3)″ of section 6. Since $c'' = a_b \parallel b_a$, by (b) of section 6, $P_a P_b = P_c = P_{c''}$ by (1) and (6) of section 6. $P_c = P_{c''} \leq P_a$, P_b by (1)″ and (3)″ of section 6. If $P_x \leq P_a, P_b$, then $x \in \mathfrak{H}_a \cap \mathfrak{H}_b$. By (6)″, $x \parallel t$ where $t \leq a$. Then $x_b \parallel t_b$, $t_b \leq a_b = c''$. Since $x \in \mathfrak{H}_b$ implies $x = x_b$, $P_x \leq P_{c''} = P_c$. This establishes P_c as $P_a \cap P_b$, the greatest lower bound (g. l. b.) of P_a and P_b. Note that on $\mathfrak{H}_b$ the mappings P_a and P_c coincide, if $c = b_a$.

If we let $c = a + b - b_a = a + b_{\perp a}$, or if we let $c' = a + b - a_b = b + a_{\perp b}$, (2)′ and (c) of section 6 show that $P_c = P_{c'} = P_a + P_r = P_b + P_s$, where $r = b_{\perp a}$ and $s = a_{\perp b}$. It is easy to prove that $P_c = P_a \cup P_b$.

$\mathfrak{P}(\mathfrak{G})$ now appears to be a lattice with universal supremum and infimum under its order relation $\leq$. Ready results are $P_a \cup P'_a = I$, $P_a \cap P'_a = 0$, so that the lattice is complemented. Moreover, $(P_a \cup P_b) \cap (P_a \cup P_c) = (P_a + P_d)(P_a + P_e)$, where $d = b_{\perp a}$ and $e = c_{\perp a}$. This product reduces to $P_a + P_f$, where $f = P_e(d)$, using (2)′ and (5)′ of section 6. Let $g = b_c$, $h = g_{\perp a}$. Then $P_a \cup (P_b \cap P_c) = P_a \cup P_g = P_a + P_h$. Now $g = P_c(b)$, $b = b_a + d$ and $c = p + e$, where $p = c_a$, $p \perp e$. Then $g = P_p(b_a) + f$, after appropriate decomposition, distribution and reduction. It follows that $h = g_{\perp a} = f_{\perp a} = f$, so that the distributive law $P_a \cup (P_b \cap P_c) = (P_a \cup P_b) \cap (P_a \cup P_c)$ is established. This proves

THEOREM 3. *Let $\mathfrak{G}$ be a* v. o. *group. Under the ordering relation $\leq$ (as described above), $\mathfrak{P}(\mathfrak{G})$, the set of projections of $\mathfrak{G}$, is a Boolean algebra.*

8. The complete algebra. In this section, we shall prove that strong $\mathfrak{G}$ leads to complete $\mathfrak{P}(\mathfrak{G})$. Even if $\mathfrak{G}$ is not strong, projections preserve l. u. b.'s of subsets of $\mathfrak{G}$, whenever these exist. For, if b is the l. u. b. of the elements of a subset $\mathfrak{B}$ of $\mathfrak{G}$, if P_a is a projection and if c is any upper bound for the elements of the set $P_a(\mathfrak{B})$, then $c_a + b_{\perp a}$ is an upper bound for the elements of $\mathfrak{B}$, by (k) of section 5. Since $c \geq c_a$ and since $c_a + b_{\perp a} \geq b$, $c_a \geq b_a$ and $c \geq c_a \geq b_a$. It follows that $b_a = P_a(b)$ is the required l. u. b. of $P_a(\mathfrak{B})$.

THEOREM 4. *Let $\mathfrak{G}$ be a* v. o. *group, and let P be the* l. u. b. *in $\mathfrak{P}(\mathfrak{G})$ of a class $\mathfrak{R}$ of projections.* (a) *If, for fixed x and z, all $R(x) \leq z$, $R \in \mathfrak{R}$, then $P(x) \in \mathfrak{H}_z$.* (b) *For each $x \in \mathfrak{G}$, $P(x)$ is minimal with respect to being an upper bound for all $R(x)$.* (c) *If, for fixed x,* l. u. b. *$R(x)$ exists, then it equals $P(x)$.*

Proof. (a) If $z = z(y) \geq R(y)$ for every $R \in \mathfrak{R}$, y fixed in $\mathfrak{G}$, then $y \geq P(y) \geq d \geq R(y)$ for every R, where $d = P_z P(y)$. Form an endomorphism $S = P_d P_y + P - PP_y$. $S(y) = d$. $S \leq P$, and S is a projection. $S(x_{\perp y}) = P(x_{\perp y}) \geq R(x_{\perp y})$ for every R. $S(x_y) = P_d(x_y)$. Let B^* denote the projection B restricted to $\mathfrak{H}_y$. Let $g = R^*(y)$ for some fixed $R \in \mathfrak{R}$; and let $h = P_y(c)$ where c is a maximal element of $\mathfrak{H}(R)$. $g, h \in \mathfrak{H}_y$. $R^* = P^*_h$, by a remark in the proof of Theorem 3. By (k) of section 6, $g \parallel h$ in $\mathfrak{H}_y$ so that $P^*_g = P^*_h$. Since $g \leq d$, $P^*_h \leq P^*_d$, and $P^*_h(x_y) \leq P^*_d(x_y) = S^*(x_y)$. But $P^*_h(x_y) = P^*_y R^*(x_y) = R^*(x_y)$. Hence $R(x_y) \leq S(x_y)$. Now $x_{\perp y} \perp x_y$, and S is a projection, so that $S(x_{\perp y}) \perp S(x_y)$. By (k) of section 5, $S(x) \geq R(x)$ for every $x \in \mathfrak{G}$. This shows that $S \geq R$ for every $R \in \mathfrak{R}$; and $S \geq P$. Since $S \leq P$, we have $S = P$. $S(y) = d$ implies $P(y) = d = P_z P(y)$, and

$P(y) \in \mathfrak{H}_z$, proving (a). The proof of (b) is immediate, and (c) follows from (b).

A similar result states that if c is the l. u. b. of a class $\mathfrak{C}$ of elements of $\mathfrak{G}$, a v. o. group, then the class $\mathfrak{P}(\mathfrak{C})$ of projections P_x, $x \in \mathfrak{C}$, has l. u. b. P_c. To prove this, we show that P_c is minimal with respect to being an upper bound for $\mathfrak{P}(\mathfrak{C})$. Since $\mathfrak{P}(\mathfrak{G})$ is a lattice, P_c is the l. u. b. for $\mathfrak{P}(\mathfrak{C})$.

We shall use the statement (e) of section 6 to prove some of the following results. Two subsets $\mathfrak{A}$ and $\mathfrak{B}$ of an ordered system $\mathfrak{G}$ are said to be order isomorphic if there is a one-to-one map f on $\mathfrak{A}$ onto $\mathfrak{B}$ such that both f and f^{-1} preserve order.

LEMMA 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be order isomorphic chains of a* v. o. *group $\mathfrak{G}$. Let $\mathfrak{C}$ be the set of all $a_b + b_a$ where $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ and a and b correspond under the order isomorphism f. Then $\mathfrak{C}$ is a chain, and there exists a (many-one) order preserving map g on $\mathfrak{A}$ onto $\mathfrak{C}$ (an order homomorphism). If $\mathfrak{A}$ and $\mathfrak{B}$ have upper bounds a^* and b^*, respectively, then $\mathfrak{C}$ has an upper bound $a^* + b^*$.*

Proof. Since $a_b \parallel b_a$, (e) of section 6 shows that $a_b \leq a'$, $b_a \leq b'$ imply $a_b + b_a \leq a' + b'$. Suppose that $a \leq r$, $a, r \in \mathfrak{A}$. By hypothesis, $f(a) = b \leq f(r)$. Denote $f(r)$ by s. $a_b \leq a_s \leq r_s$. Let $b_a \leq s_r$. Taking $a' = b_s$, $b' = s_r$, we have $a_b + b_a \leq r_s + s_r$. Define the mapping g by $g(a) = a_b + b_a$ where $f(a) = b$. The above shows that $a \leq r$ implies $g(a) \leq g(b)$. The remaining statements of the lemma follow directly.

LEMMA 3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be order isomorphic chains of a* v. o. *group $\mathfrak{G}$. Let $\mathfrak{C}$ be the set of all a_b where $a \in \mathfrak{A}$, $b \in \mathfrak{B}$ and a and b correspond under the isomorphism: $b = f(a)$. Then $\mathfrak{C}$ is a chain and is an order homomorph of $\mathfrak{A}$ under a mapping g. Let a^* and b^* be the respective* l. u. b.*'s of $\mathfrak{A}$ and $\mathfrak{B}$. If w is any upper bound for $\mathfrak{C}$, then $a^*_{b^*} \in \mathfrak{H}_w$. $a^*_{b^*}$ is an upper bound for $\mathfrak{C}$ and is its* l. u. b. *if $\mathfrak{C}$ has a* l. u. b. *c^*.*

Proof. That $\mathfrak{C}$ is a chain, that g exists with $g(a) = a_b$, $b = f(a)$, and that $a^*_{b^*}$ is an upper bound for $\mathfrak{C}$ are all evident. Suppose that w is an upper bound for all $a_b \in \mathfrak{C}$. Since $a^* =$ l. u. b. a, $a \in \mathfrak{A}$, and since projections preserve l. u. b.'s, $a^*_b =$ l. u. b. a_b, b fixed in $\mathfrak{B}$. Hence $w \geq a^*_b$ for all $b \in \mathfrak{B}$. Now, $b^* =$ l. u. b. b, $b \in \mathfrak{B}$ implies $P_{b^*} =$ l. u. b. P_b. By Theorem 4(a), $P_{b^*}(a^*) = a^*_{b^*} \in \mathfrak{H}_w$. If $\mathfrak{C}$ has l. u. b. c^*, then $a^*_{b^*} \geq c^*$ and $a^*_{b^*} \in \mathfrak{H}_{c^*}$. Thus, $P_{c^*}(a^*_{b^*}) = c^* = a^*_{b^*}$.

LEMMA 4. (a) *Let $\mathfrak{A}$ and $\mathfrak{B}$ be subsets of $\mathfrak{G}$, a* v. o. *group, and let $\mathfrak{A}$ and $\mathfrak{B}$ be in one-to-one correspondance via a mapping f. Suppose that $a \parallel f(a)$*

for every $a \in \mathfrak{A}$ *and that* $a, f(a) \in \mathfrak{H}_d$, *where* $d = a + f(a)$, $d = d(a)$. *If all three quantities exist,* l. u. b. $a +$ l. u. b. $f(a) =$ l. u. b. $d(a)$. (b) *For* $x, y \in \mathfrak{G}$ *suppose that* $x \parallel y \parallel x + y$ *and that* $\mathfrak{R}$ *is a set of projections on* $\mathfrak{G}$. *If the three quantities exist,* l. u. b. $P(x) +$ l. u. b. $P(y) =$ l. u. b. $P(x + y)$, *where the* l. u. b. *are taken over all* $P \in \mathfrak{R}$. (c) *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be subsets of* v. o. $\mathfrak{G}$, *where* $\mathfrak{A} \perp \mathfrak{B}$. *Let* $\mathfrak{C}$ *be* $\mathfrak{A} + \mathfrak{B}$, *the addition being that of complexes. Let* $\mathfrak{A}$, $\mathfrak{B}$ *and* $\mathfrak{C}$ *have the respective* l. u. b.'s a^*, b^* *and* c^*. *Then* $a^* \perp b^*$, *and* $c^* = a^* + b^*$. (d) *Suppose that* $x \perp y$ *in* $\mathfrak{G}$ *and that* $\mathfrak{C}$ *is a chain of projections on* $\mathfrak{G}$. *Suppose that* l. u. b. $P(x)$, l. u. b. $P(y)$ *and l. u. b.* $P(x + y)$ *exist, where the* l. u. b. *are taken over all* $P \in \mathfrak{C}$. *Then* l. u. b. $P(x + y) =$ l. u. b. $P(x) +$ l. u. b. $P(y)$, *and* l. u. b. $P(x) \perp$ l. u. b. $P(y)$.

Proof. (b) is a corollary of (a) and (d) is a corollary of (c). (a) and (c) have uncomplicated proofs.

THEOREM 5. *The set* $\mathfrak{P}(\mathfrak{G})$ *of projections of a strong vector ordered group* $\mathfrak{G}$ *forms, under its natural order, a complete Boolean algebra.*

Proof. We have already proved that $\mathfrak{P}(\mathfrak{G})$ is a Boolean algebra. Let $\mathcal{A}$ be a subset of $\mathfrak{P}(\mathfrak{G})$. P_0 is a lower bound for the elements of $\mathcal{A}$. Let $\mathcal{B}$ be the non-empty set of lower bounds of $\mathcal{A}$ in $\mathfrak{P}(\mathfrak{G})$. If every chain in $\mathfrak{P}(\mathfrak{G})$ were to have a l. u. b. in $\mathfrak{P}(\mathfrak{G})$, then Zorn's Lemma would provide a maximal element P^* of $\mathcal{B}$. It is easy to see that P^* would then be the required g. l. b. for $\mathcal{A}$, and we could complete the proof by complementation.

Hence let $\mathcal{C}$ be a chain of projections. For $x \in \mathfrak{G}$ define $P^*(x)$ as l. u. b. $P(x)$, the l. u. b. taken over all $P \in \mathcal{C}$. We shall prove that P^* is a projection of $\mathfrak{G}$ and that $P^* =$ l. u. b. P, $P \in \mathcal{C}$. It is clear that $P^*(0) = 0$ and that $P^*(-x) = -P^*(x)$. For $x, y \in \mathfrak{G}$, set $x_y = s$, $y_x = t$. The three quantities $x_{\perp y}$, $y_{\perp s}$ and $s + t$ are mutually $\perp$. By Lemma 4(d), $P^*(x + y) = P^*(x_{\perp y}) + P^*(y_{\perp x}) + P^*(s + t)$. But $s \parallel t$ and $P_{s+t}(s) \parallel P_{s+t}(t)$, where the sum of the two latter quantities is $s + t$. By Lemma 4(b), $P^*(s + t) = P^*P_{s+t}(s) + P^*P_{s+t}(t)$. The triad $x_{\perp y}$, $P_{s+t}(s)$ and $P'_{s+t}(s)$ consists of mutually $\perp$ elements; likewise for $y_{\perp x}$, $P_{s+t}(t)$ and $P'_{s+t}(t)$. Now $x = x_{\perp y} + P_{s+t}(s) + P'_{s+t}(s)$, and $y = y_{\perp x} + P_{s+t}(t) + P'_{s+t}(t)$. By Lemma 4(d)

$$P^*(x) + P^*(y) = P^*(x_{\perp y}) + P^*(y_{\perp x}) + P^*P_{s+t}(s) + P^*P_{s+t}(t) + P^*P'_{s+t}(s) + P^*P'_{s+t}(t).$$

But $P'_{s+t}(s) + P'_{s+t}(t) = P'_{s+t}(s + t) = 0$. Since $P^*(-z) = -P^*(z)$, $P^*P'_{s+t}(s) + P^*P'_{s+t}(t) = 0$. Combining results, $P^*(x + y) = P^*(x) + P^*(y)$, and P^* is established as an endomorphism. That $P^* =$ l. u. b. P and that

$P^* \leq I$ are quite trivial. Hence P^* is a projection with the required property, and the proof of the theorem is complete.

THEOREM 6. *A strong* v. o. *group* $\mathfrak{G}$ *is upper conditionally complete and lower complete.*

Proof. The lower completeness is the statement of $6(c')$ of Lemma 1. Let $\mathcal{A}$ be a set of projections of $\mathfrak{G}$. These have a l. u. b. P^*, by Theorem 5. By Theorem 4(b), $P^*(x)$ is a minimal upper bound for the $P(x)$, $P \in \mathcal{A}$, x fixed in $\mathfrak{G}$. The set $\mathfrak{M}$ of such minimal upper bounds is non-empty and has a g. l. b. t which is a upper bound for the $P(x)$. By the minimal character of $P^*(x)$, $t = P^*(x)$ so that $P^*(x) =$ l. u. b. $P(x)$. Now let $\mathfrak{C}$ be a class of elements of $\mathfrak{G}$, and let $c^\#$ be an upper bound for the elements of $\mathfrak{C}$. Let $P^* =$ l. u. b. P_c, $c \in \mathfrak{C}$. In the above, let $x = c^\#$. Then $P^*(c^\#) =$ l. u. b. $P_c(c^\#)$. But $P_c(c^\#) = c$, so that l. u. b. c, $c \in \mathfrak{C}$ exists and equals $P^*(c^\#)$.

We have two immediate corollaries.

COROLLARY 1. $[x \mid x \in \mathfrak{G}; x \leq a]$, *(where* a *is fixed in strong* v. o. $\mathfrak{G}$*), is, under the order relation hereditary from* $\mathfrak{G}$*, a complete Boolean algebra isomorphic to* $[P_x \mid x \in \mathfrak{G}; P_x \leq P_a]$*, where the order relation in the latter set is hereditary from* $\mathfrak{P}(\mathfrak{G})$.

COROLLARY 2. *Let* $\mathbf{K}$ *be any set covering of strong* v. o. $\mathfrak{G}$*, and let* $\mathfrak{H}$ *be any subset of* $\mathfrak{G}$*. Let* g. l. b. $\mathfrak{H}$ *have the obvious meaning. Then* g. l. b. $\mathfrak{H}$ $=$ g. l. b. (g. l. b. $\mathfrak{H} \cap \mathcal{U}$) *where the* $\mathcal{U}$ *are the sets of* $\mathbf{K}$.

Part II.

9. Boolean groups. Let $\mathfrak{B}$ be a Boolean algebra with ordering relation $\leq$ and universal supremum I. A partition Π of I is a collection of elements of $\mathfrak{B}$ such that (1) α, $\beta \in \Pi$, $\alpha \neq \beta$ imply $\alpha \perp \beta$, and (2) l. u. b. $\alpha = I$, the l. u. b. taken over all $\alpha \in \Pi$. We exclude repetitions from a partition and also the element 0, the universal infimum of $\mathfrak{B}$. Π' is said to be a refinement of Π, $(\Pi' \prec \Pi)$, if to each $\alpha \in \Pi'$ there exists $\beta(\alpha) \in \Pi$ with $\alpha \leq \beta(\alpha)$. Given two partitions of I, Π and Π', there exists a common refinement $\Pi'' = [\alpha \cap \beta \mid \alpha \in \Pi, \beta \in \Pi']$, as the infinite distributive law for Boolean algebras shows [2, p. 165]. Call this the intersection refinement of Π and Π'.

Let Σ be a fixed, non-trivial additive Abelian group (not necessarily ordered in any way). This will be called the group of coefficients. To each $\beta \in \Pi$, a partition, attach a coefficient $s(\beta) \in \Sigma$. Designate the new object thus formed by (Π, s), where s is the function $\beta \to s(\beta)$. (Π, s) is to be

called a valued partion (of I in $\mathfrak{B}$). We can say that (Π', s') is a refinement of $(\prec)$ (Π, s) if (1) $\Pi' \prec \Pi$ and (2) $\alpha \in \Pi$, $\beta \in \Pi$, $\alpha \leq \beta$ imply $s(\beta) = s'(\alpha)$. We can define $(\Pi, s) \sim (\Pi', s')$ if these two valued partitions have a common refinement. If $(\Pi, s) \sim (\Pi, s')$, then $s = s'$. Given (Π, s) and $\Pi' \prec \Pi$, there exists s' with $(\Pi, s) \sim (\Pi', s')$. The relation $\sim$ is an equivalence relation on the set of valued partitions of I of $\mathfrak{B}$. Symmetry and reflexivity are trivial. As for transitivity, suppose that $(\Pi_1, s_1) \sim (\Pi_2, s_2)$, $(\Pi_2, s_2) \sim (\Pi_3, s_3)$ with respective common refinements (Π_{12}, s_{12}) and (Π_{23}, s_{23}). Let Π be the intersection refinement of Π_{12} and Π_{23}. A given $\alpha \in \Pi$ equals $\beta_1 \cap \beta_3$, $\beta_1 \in \Pi_{12}$, $\beta_3 \in \Pi_{23}$. Also there exist $\gamma_{1i} \in \Pi_i$, $i = 1, 2$, and $\gamma_{3j} \in \Pi_j$, $j = 2, 3$, with $\beta_1 \leq \gamma_{11}, \gamma_{12}$ and $\beta_3 \leq \gamma_{32}, \gamma_{33}$. Hence $\alpha \leq \gamma_{12} \cap \gamma_{32}$, so that the latter is not 0. Hence $\gamma_{12} = \gamma_{32}$ since Π_2 can have no repetitions and no overlapping elements. Define a mapping s on Π into Σ by $s(\alpha) = s_2(\gamma_{12})$ $(= s_2(\gamma_{32}))$. Then $(\Pi, s) \prec (\Pi_1, s_1)$, (Π_3, s_3) and the transitivity is established.

We denote the set of equivalence classes under $\sim$ by $\mathfrak{G}(\mathfrak{B}, \Sigma)$ or simply by $\mathfrak{G}$, if no confusion arises. Each equivalence class $g \in \mathfrak{G}$ can be denoted by $\{(\Pi, s)\}$, and we say that $(\Pi, s) \in g$, or that g is represented by a valued partition (Π, s) (and by all equivalent valued partitions). We can introduce an addition in $\mathfrak{G}$ quite simply. If $(\Pi_1, s_1) \in g$, $(\Pi_2, s_2) \in h$, let Π_{12} be the intersection refinement of Π_1 and Π_2, and define s_{12} on Π_{12} into Σ by $s_{12}(\alpha \cap \beta) = s_1(\alpha) + s_2(\beta)$, $\alpha \in \Pi_1$, $\beta \in \Pi_2$. Suppose that $(\Pi_0, s_0) \in \{(\Pi_1, s_1)\} = g$. From (Π_0, s_0) and (Π_2, s_2) form (Π_{02}, s_{02}) by the above addition process. Construct a common refinement (Π, s) of (Π_0, s_0) and (Π_1, s_1). Apply the above addition process to (Π, s) and (Π_2, s_2), obtaining (Π', s'). It is easily seen that (Π', s') is a common refinement of (Π_{12}, s_{12}) and (Π_{02}, s_{02}), so that addition is independent of the members of the equivalence classes which are used as summands. The associativity of addition in $\mathfrak{G}$ follows from that in Σ; and the 0 and the additive inverses can be defined in the obvious way. No confusion will arise if we still denote by $\mathfrak{G}$ the set $\mathfrak{G}$ with this definition of addition. We have proved that $\mathfrak{G}$ is a group, in fact, an Abelian group. We shall call such groups Boolean groups.

$\mathfrak{G}$, a Boolean group, can be partly ordered as follows. For $g, h \in \mathfrak{G}$, $g \leq h$ if there exist $(\Pi, s_1) \in g$, $(\Pi, s_2) \in h$, with $s_1(\alpha) = 0$ or $s_2(\alpha)$ for every $\alpha \in \Pi$. We call (Π, s_1) and (Π, s_2) $g \leq h$ representatives. If Π' is a refinement of Π, it follows readily that there exist s'_1 and s'_2 such that $(\Pi', s'_1) \prec (\Pi, s_1)$, $(\Pi', s'_2) \prec (\Pi, s_2)$ and (Π', s'_1) and (Π', s'_2) are $g \leq h$ representatives. The following likewise has an immediate proof. If $\mathfrak{B}$ is a complete Boolean algebra, if $(\Pi_1, s_1) \sim (\Pi_2, s_2)$ and if $\sigma \in \Sigma$, then l. u. b. α, $\alpha \in \Pi_1$, $s_1(\alpha) = \sigma$, equals l. u. b. β, $\beta \in \Pi_2$, $s_2(\beta) = \sigma$.

THEOREM 7. *Under the relation* $\leq$, *every Boolean group* $\mathfrak{G}$ *is vector ordered.*

Proof. (a) $g \leq g$, for every $g \in \mathfrak{G}$. (b) If $x \leq y$, $y \leq z$, we can, by the above remarks, choose $x \leq y$ and $y \leq z$ representatives so that the representation (Π, s) for y is the same in both cases. It follows at once that $x \leq z$. (c) If $x \leq y$, $y \leq x$, represent y by (Π, s) in both inequalities. This gives x two representations, $(\Pi, s_1) \in x$, where $x \leq y$, and $(\Pi, s_2) \in x$, where $y \leq x$. Hence, $s_1 = s_2$. But $s_1(\alpha) = 0$ or $s(\alpha)$, and $s(\alpha) = 0$ or $s_2(\alpha)$. Since $s_2(\alpha) = s_1(\alpha)$, $s = s_1$ and $x = y$. This establishes $\mathfrak{G}$ as a partly ordered system.

We verify the axioms for v. o. groups in order. (1) is trivial. (2) Let $x \geq y$ representatives be $(\Pi, s_2) \in x$, $(\Pi, s_1) \in y$. Then $x - y = \{(\Pi, s_2 - s_1)\}$. Since $(s_2 - s_1)(\alpha) = 0$ or $s_2(\alpha)$, for every $\alpha \in \Pi$, $x \geq x - y$. If $z \leq y$, $x - y$, we can redefine Π and find s_3, s'_1 and s^* such that $(\Pi, s_3) \in z$, $(\Pi, s'_1) \in y$, $(\Pi, s^*) \in x - y$. Also, there are the obvious relations between s_3 and s'_1 and between s_3 and s^*. $s_3(\alpha) \neq 0$ implies $s'_1(\alpha) = s^*(\alpha) = s_3(\alpha)$. With the redefinition of Π, $s_1 = s'_1$, and $s^* = s_2 - s_1$. Hence $s_1(\alpha) \neq 0$, and $x \geq y$ implies $s_2(\alpha) = s_1(\alpha)$. This makes $s^*(\alpha) = 0$, a contradiction. It follows that $s_3(\alpha) = 0$ for every $\alpha \in \Pi$, and $z = 0$. We have proved that if $x \geq y$, then $y \perp x - y$.

(3) Suppose that $(\Pi, s_1) \in x$, $(\Pi, s_2) \in y$, $(\Pi, s_3) \in z$ and that $w \geq x, y$ where $x \perp y$. The standard relations are to hold between s_1 and s_3 and between s_2 and s_3. If $s_1(\beta)$ and $s_2(\beta)$ are both non-zero, then $s_1(\beta) = s_2(\beta) = s_3(\beta)$. Let $s(\alpha) = 0$ if $\alpha \in \Pi$, $\alpha \neq \beta$, and let $s(\beta) = s_3(\beta)$. $z = \{(\Pi, s)\} \neq 0$, and $z \leq x, y$. Since $x \perp y$, $z = 0$, a contradiction, and not both $s_1(\beta)$ and $s_2(\beta)$ can be non-zero. $x + y = \{(\Pi, s_1 + s_2)\}$; but $(s_1 + s_2)(\alpha) = s_1(\alpha)$ if the latter is non-zero, $s_2(\alpha)$ otherwise. This is enough to show that $x, y \leq x + y$ if $x \perp y$.

(4) Represent x, y and w as in (3) where, this time, $x + y$, $w \geq x, y$. As in (3), not both $s_1(\alpha)$ and $s_2(\alpha)$ can be non-zero, and we can arrange things so that $s_1(\alpha)$ and $s_2(\alpha)$ are 0 or $s_1(\alpha) + s_2(\alpha)$. If $s_2(\alpha) \neq 0$, $s_1(\alpha) + s_2(\alpha) = s_2(\alpha) = s_3(\alpha)$. If $s_2(\alpha) = 0$, $s_1(\alpha) + s_2(\alpha) = s_1(\alpha)$ which is 0 or $s_3(\alpha)$. This shows that $x + y \leq w$. Therefore, if $x + y \geq x, y$, then $x + y = x \cup y$.

(5) (a) If $(\Pi, s_1) \in x$, $(\Pi, s) \in a$, $x \perp a$, then $s_1(\alpha)$ and $s(\alpha)$ cannot both be non-zero, as in (3) above. $(\Pi, -s_1) \in -x$, where $-s_1$ is defined in the obvious way. $(\Pi, s - s_1) \in a - x$. But $(s - s_1)(\alpha) = s(\alpha)$ if the latter is non-zero, $-s_1(\alpha)$ otherwise. Therefore $a \leq a - x$. By (2) above, $a \perp -x$.

If $x, y \perp a$, a similar procedure shows that $a \leq a + x + y$. Applying (2), $a \perp x + y$. Hence $\mathfrak{K}_a$ is a subgroup of $\mathfrak{G}$.

(5)(b) We can use (2), (5)(a) and (3) to show that $x \in \mathfrak{H}_a$ implies $-x \in \mathfrak{H}_a$. Write, now, $(\Pi, s_1) \in x$, $(\Pi, s_2) \in y$, $(\Pi, s) \in a$, $(\Pi, s') \in u$ where $u \leq x + y$, $u \perp a$ and where $x, y \in \mathfrak{H}_a$. As above, $s(\alpha)$ and $s'(\alpha)$ cannot both be non-zero. If $s'(\beta) \neq 0$ for some $\beta \in \Pi$, then $s(\beta) = 0$ and not both $s_1(\beta)$ and $s_2(\beta)$ are 0, say, $s_1(\beta) \neq 0$. Define t by $t(\alpha) = 0$ if $\alpha \in \Pi$, $\alpha \neq \beta$, and $t(\beta) = s_1(\beta)$. Let $u' = \{(\Pi, t)\}$. $u' \leq x$, $u' \neq 0$. $t(\beta) = s_1(\beta) \neq 0$, while $s(\beta) = 0$. t and s never assume non-zero values together over Π. Hence $u' + a \geq u', a$, so that, by (2), $u' \perp a$. Since $x \in \mathfrak{H}_a$, $u' = 0$, a contradiction. Thus $s'(\beta) = 0$ for every $\beta \in \Pi$, and $u = 0$. This proves that $\mathfrak{H}_a$ is a subgroup of $\mathfrak{G}$.

(6)(a) If $x = \{(\Pi, s)\}$ and if x is not maximal, there exists at least one $\alpha \in \Pi$ for which $s(\alpha) = 0$. Fix $\sigma \in \Sigma$, $\sigma \neq 0$. Define t by $t(\beta) = s(\beta)$ if $s(\beta) \neq 0$, $t(\beta) = \sigma$ otherwise. Let $m = \{(\Pi, t)\}$. Then m is maximal in $\mathfrak{G}$ and $m \geq x$.

(6)(b) If $x = \{(\Pi, s_1)\}$, $y = \{(\Pi, s_2)\}$, define s by $s(\alpha) = s_1(\alpha)$ if $s_2(\alpha) = 0$, and $s(\alpha) = 0$ otherwise. If we take $z = \{(\Pi, s)\}$, a direct proof shows that $z = x_{\perp y}$.

(6)(c) For x, y as in (6)(b), define t by $t(\alpha) = s_1(\alpha)$ if $s_1(\alpha) = s_2(\alpha)$, and $t(\alpha) = 0$ otherwise. Let $w = \{(\Pi, t)\}$. Then a simple check shows that $w = xy$, the g. l. b. of x and y.

10. Cardinal restrictions on partitions. In section 9, no restrictions were placed on the number of elements. However, if we demand that the power of the class of elements in a given partition Π is $< \aleph$ (or $\leq \aleph$), a fixed cardinal, none of the proofs of section 9 is materially altered. In particular, we can restrict partions to have only a finite collection of elements.

If Σ is a ring, $\mathfrak{G}(\mathfrak{B}, \Sigma)$ will also be a ring if we introduce multiplication into $\mathfrak{G}$ in the same manner as addition was introduced. Since the term Boolean ring is well established for another sort of ring, we shall call the present ring a Boolean type ring. It will turn out that every Boolean ring with a unity is also a Boolean type ring. We return to these rings later.

THEOREM 8. *If $\mathfrak{B}$ is a complete Boolean algebra, then $\mathfrak{G}(\mathfrak{B}, \Sigma)$ is a strong* v. o. *group.*

Proof. We make no restriction on the cardinals of the partitions Π. Let $\mathfrak{C}$ be a chain of elements of $\mathfrak{G}$. For each non-zero $\sigma \in \Sigma$, construct $\beta(\sigma)$, the union (l. u. b.) of all elements which are valued with σ in all partitions

Π which are used to represent the elements of $\mathscr{C}$. Let $\beta(0)$ be the complement in $\mathscr{B}$ of the union of all the $\beta(\sigma)$. $\beta(0) \cap \beta(\sigma) = 0$ for every $\sigma \in \Sigma$, $\sigma \neq 0$. Suppose that $\beta(\sigma) \cap \beta(\tau) \neq 0$, where σ and τ are distinct, non-zero elements of Σ. Then there exist two distinct members of the chain $\mathscr{C}$, say, $x \leq y$, $x = \{(\Pi, s_1)\}$, $y = \{(\Pi, s_2)\}$, and there exists $\alpha \in \Pi$ with $s_1(\alpha) = \sigma$, $s_2(\alpha) = \tau$. But $x \leq y$ and $s_1(\alpha) = \sigma \neq 0$ imply $s_2(\alpha) = s_1(\alpha)$ or that $\tau = \sigma$, a contradiction. Thus $\beta(\sigma) \cap \beta(\tau) = 0$ for every pair of unequal σ and τ. On the other hand, the l. u. b. of all the $\beta(\sigma)$ is I, the supremum of $\mathscr{B}$, and the $\beta(\sigma)$ form a partition of I. Call this partition Ω. Define t by $t(\beta(\sigma)) = \sigma$, a mapping on Ω into Σ, and let $x^* = \{(\Omega, t)\}$. It is clear that x^* is the l. u. b. of the elements of $\mathscr{C}$.

11. The representation theorems. Let $\mathscr{B}$ be a Boolean algebra and let Σ, the coefficient group, be non-trivial. Fix $\beta \in \mathscr{B}$ and agree to use only those partitions of I which (1) have only a finite number of elements and (2) are refinements of the partition $\{\beta, \beta'\}$, where β' is the complement of β in $\mathscr{B}$. For $x = \{(\Pi, s)\}$, let $x(\beta) = \{(\Pi, s')\}$, where $s'(\alpha) = s(\alpha)$ if $\alpha \leq \beta$ and $s'(\alpha) = 0$ if $\alpha \leq \beta'$, $\alpha \in \Pi$. It follows readily that the mapping P_β: $x \to x(\beta)$ is a projection of $\mathfrak{G}(\mathscr{B}, \Sigma)$. Define a mapping Φ on $\mathscr{B}$ into $\mathfrak{P}(\mathfrak{G}(\mathscr{B}, \Sigma))$ by $\Phi(\beta) = P_\beta$. If $P_a = P_\beta$, then $x(\alpha) = x(\beta)$ for every $x \in \mathfrak{G}$. If $\gamma \leq \alpha \cap \beta'$, then $x(\gamma) \leq x(\alpha), x(\beta')$. Since $x(\beta') \perp x(\beta)$, $x(\gamma) = 0$ for every $x \in \mathfrak{G}$. But this is impossible unless $\gamma = 0$. Therefore $\alpha \cap \beta' = 0$, $\alpha \leq \beta$ and similarly $\beta \leq \alpha$. This shows that Φ is one-to-one. If $P \in \mathfrak{P}$, then $P = P_x$, $x \in \mathfrak{G}$. Let $x = \{(\Pi, s)\}$. Here, by (1), Π has only a finite number of elements. If $x = 0$, $P = P_0 = P_\beta$, where $\beta \in \mathscr{B}$, $\beta = 0$, the infinum of $\mathscr{B}$. If $x \neq 0$, let $\beta =$ l. u. b. α, $\alpha \in \Pi$, $s(\alpha) \neq 0$. Then $P = P_x = P_\beta$. We have proved that Φ is onto $\mathfrak{P}$. Now $\alpha \cup \beta = (\alpha \cap \beta) \cup (\alpha \cap \beta') \cup (\alpha' \cap \beta)$, the l. u. b. of three disjoint quantities. As endomorphisms of $\mathfrak{G}$,

$$\begin{aligned}\Phi(\alpha \cup \beta) &= \Phi(\alpha \cap \beta) + \Phi(\alpha \cap \beta') + \Phi(\alpha' \cap \beta) \\ &= \Phi(\alpha \cap \beta) \cup \Phi(\alpha \cap \beta') \cup \Phi(\alpha' \cap \beta) \cup \Phi(\alpha \cap \beta) \\ &= \Phi(\alpha) \cup \Phi(\beta).\end{aligned}$$

For, the Φ's are elements of $\mathfrak{P}$, and in $\mathfrak{P}$, $P \perp Q$ implies $P + Q = P \cup Q$. Moreover, $\alpha = (\alpha \cap \beta) \cup (\alpha \cap \beta')$ and $\beta = (\alpha \cap \beta) \cup (\alpha' \cap \beta)$. As for $\cap$, $\Phi(\alpha') = P_{a'} = P'_a$, as defined in section 6, and P'_a is the complement of P_a in $\mathfrak{P}$, by section 7. Thus $\Phi(\alpha \cap \beta) = \Phi((\alpha' \cup \beta')') = P'_{a' \cup \beta'}$ $= [P_{a'} \cup P_{\beta'}]' = P'_{a'} \cap P'_{\beta'} = P_a \cap P_\beta = \Phi(\alpha) \cap \Phi(\beta)$. We have proved

THEOREM 9. *A Boolean algebra $\mathscr{B}$ has a faithful (lattice isomorphic)*

representation as the set of all projections on a Boolean group (therefore on a vector ordered group) $\mathfrak{G}(\mathfrak{B}, \Sigma)$.

Now let Σ be a ring so that $\mathfrak{G}$ is a Boolean type ring. A projection P of the group structure of $\mathfrak{G}$ will preserve products so that it is a ring endomorphism. In Theorem 9, we can replace the words "Boolean group" by "Boolean type ring." In particular, let Σ be the ring $\mathfrak{J}_2$ of integers modulo 2. Let us define a map θ on $\mathfrak{B}$ into $\mathfrak{G}(\mathfrak{B}, \mathfrak{J}_2)$ such that $\theta(\alpha) = \{(\Pi, s)\}$, where Π is the partition $\{\alpha, \alpha'\}$ and $s(\alpha) = 1_2$, $s(\alpha') = 0_2$. Note that the product in $\mathfrak{G}$ (componentwise multiplication) coincides with the product of axiom (6)(c). We restrict all partitions to those with only a finite number of elements. Then θ turns out to be one-to-one onto $\mathfrak{G}$. $\theta(\alpha \cap \beta) = \theta(\alpha)\theta(\beta)$, where the product on the right is the ring product in $\mathfrak{G}$. Let $+_2$ denote the symmetric difference in $\mathfrak{B}$. Then $\theta(\alpha +_2 \beta) = \theta((\alpha \cap \beta') \cup (\alpha' \cap \beta)) = \theta(\alpha \cap \beta') + \theta(\alpha' \cap \beta)$ since $\alpha \cap \beta' \perp \alpha' \cap \beta$. We can add in $0 = 2\theta(\alpha \cap \beta)$ on the right hand side. Then we see that $\theta(\alpha +_2 \beta) = \theta(\alpha) + \theta(\beta)$. We thus have

THEOREM 10. *Every Boolean algebra has a faithful representation as a set of splitting endomorphisms on a Boolean ring with a unity, namely on itself as a Boolean ring, and every Boolean ring with a unity is a Boolean type ring.*

Finally, if $\mathfrak{B}$ is a complete Boolean algebra, we can drop the restriction that each partition Π have only a finite number of elements. Then $\mathfrak{B}$ will be faithfully represented by the set of projections on a strong v. o. group.

WASHINGTON UNIVERSITY IN SAINT LOUIS.

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DIRECT DECOMPOSITIONS IN LATTICES.*

By L. Aileen Hostinsky.

1. Introduction. A comprehensive refinement theorem for direct decompositions of operator loops is proved in [1]. In this paper we give a generalization of that theorem by using the results of a previous paper (see [4]) to translate into lattice theory Fitting's method of investigating the refinement theorems. Thus we define direct decompositions by their decomposition endomorphisms and use products of these decomposition endomorphisms as a principal tool. The definition of endomorphism and the postulates imposed upon the lattice considered are those of [4].

Although previously it had been possible to state assertions in the refinement theorems in terms of the lattice only, the hypotheses of many theorems dealt also with other terminology. This fact is true particularly in those cases where it was attempted to use Fitting's method exclusively. Because of the results in [4], this method can now be used in the lattice also. There are, however, major differences in the proofs since no use can be made here of the elements (in the sense of [1]) and since pairs of complementary decomposition endomorphisms δ, ϵ do not have the property that $\delta + \epsilon = 1$, which was much used previously.

Except for a slight modification of the Splitting Hypothesis, our refinement theorems are exactly analogous to those of [1].

Splitting Hypothesis. *If δ, ϵ is a pair of complementary decomposition endomorphisms of $p/0$ and if α is a decomposition endomorphism of $p/0$, then $\alpha\delta\alpha\epsilon\alpha$ is a uniformly splitting endomorphism of $p/0$.*

Special Refinement Theorem. *If the Splitting Hypothesis is satisfied by p and if $p = a \oplus b = d \oplus e$, then there exist direct decompositions*

$$a = a' \oplus a'', \quad b = b' \oplus b'', \quad d = d' \oplus d'', \quad \text{and} \quad e = e' \oplus e''$$

such that

$$a' \oplus b' = b' \oplus d' = d' \oplus e' = e' \oplus a' \quad \text{and}$$
$$a'' \oplus b'' = b'' \oplus e'' = e'' \oplus d'' = d'' \oplus a''.$$

Definition. *The permutation $i \to i^*$ of the integers from 1 to n is*

* Received March 31, 1950.

said to effect an exchange isomorphism of the direct decomposition $p = a(1) \oplus \cdots \oplus a(n)$ *upon the direct decomposition* $p = b(1) \oplus \cdots \oplus b(n)$ *if for every* i

$$p = a(1) \oplus \cdots \oplus a(i-1) \oplus b(i^*) \oplus a(i+1) \oplus \cdots \oplus a(n).$$

GENERAL REFINEMENT THEOREM. *If the splitting hypothesis is satisfied by* p, *then any two direct decompositions of* p *possess exchange-isomorphic refinements.*

The last section of the paper is devoted to showing the relationship between our results and those of Ore and Kurosch, who previously have dealt with the problem of direct decompositions in Dedekind structures. Through translation into terms of homomorphism and decomposition endomorphism, concepts and statements of [7] are made more concise, and essentially his main theorem is seen to be contained in ours.

The problem of this paper was suggested by Professor Reinhold Baer, whose valuable suggestions the author gratefully acknowledges.

For the convenience of the reader the remaining part of this section consists of a statement of the postulates and those definitions and theorems of [4] which are used here.

The algebraic system considered is a complete modular lattice P upon which is imposed one additional postulate. The elements of P are denoted by lower-case italic letters and the partial ordering by $\leq$. If p and q are elements in P, there exist (uniquely determined) elements pq, the *product* of p and q, and $p+q$, the *sum* of p and q, such that $pq \leq p \leq p+q$, $pq \leq q \leq p+q$ and such that $b \leq p \leq a$ and $b \leq q \leq a$ together imply that $b \leq pq$ and $p+q \leq a$. In addition to Dedekind's modular law, we assume also the existence of an element 0 in P such that $0 \leq p$ for every p in P and the following additional property:

POSTULATE (*). *If* a *and* $\{p_\nu\}$, *where* ν *ranges over a set of ordinals, are elements in* P *and if the* p_ν *form an ascending chain, then* $a \sum_\nu p_\nu = \sum_\nu a p_\nu$.

A single-valued transformation η is a *homomorphism* of p/p' in P upon q/q' in Q (a system satisfying the same postulates as P, or P itself) if η maps x/p' for every x satisfying $p' \leq x \leq p$ upon $x\eta/q'$, where $q' \leq x\eta \leq q$, so that the following requirements hold:

1) $(\sum_x x)\eta = \sum_x (x\eta)$ (additivity).

2) If $q' \leq z \leq x\eta$, then there exists an element z' with $p' \leq z' \leq x$ such that $z'\eta = z$ (exhaustiveness).

3) If $x\eta = y\eta$, then there exist elements x' and y' ($p' \leq x' \leq p$ and $p' \leq y' \leq p$) such that $x + x' = y + y'$ and $x'\eta = y'\eta = q'$.

The *kernel* $k(\eta)$ of a mapping η of p/p' on q/q' is the sum of all elements which η maps upon q'. If η is a homomorphism of p/p' upon q/q' and if $p' \leq x \leq p$, then

1) $x\eta = q'$ if and only if $x \leq k(\eta)$.

2) $x\eta = y\eta$ if and only if $x + k(\eta) = y + k(\eta)$.

An *isomorphism* of p/p' upon q/q' is a one-to-one, exhaustive, and additive mapping of p/p' upon q/q'. An isomorphism η of $p/0$ upon $q/0$ has the property $p\eta = q$. An *automorphism* of p/p' is an isomorphism of p/p' upon itself. A homomorphism η of p/p' upon q/q' is an isomorphism if and only if $k(\eta) = p'$. Moreover, products of homomorphisms (isomorphisms) are homomorphisms (isomorphisms).

The mapping η is an *endomorphism* of $p/0$ if $p\eta \leq p$ and if η is a homomorphism of $p/0$ upon $p\eta/0$. The kernels $k(\eta^i)$, for $i = 1, 2, \cdots$, of the endomorphism η^i of $p/0$ form an ascending chain $k(\eta) \leq k(\eta^2) \leq \cdots \leq p$, and the *radical* of the endomorphism η, denoted by $r(\eta)$, is defined to be the sum $\sum_{i=1}^{\infty} k(\eta^i)$.

THEOREM A. *If η is an endomorphism of $p/0$, then*

1) *If $x \leq p$, $x \leq r(\eta)$ if and only if $x\eta \leq r(\eta)$.*

2) *η induces an endomorphism of $r(\eta)/0$.*

If η is an endomorphism of $p/0$ and if $x \leq p$ satisfies $x\eta \leq x$, x is *η-admissible*. If η is an endomorphism of $p/0$ and if x is η-admissible, then η induces in $x/0$ an endomorphism of kernel $xk(\eta)$ and of radical $xr(\eta)$.

The element c is a *complement* of the endomorphism η of $p/0$ if $p = r(\eta) + c$, $cr(\eta) = 0$, and $c\eta = c$. These first two conditions state that p is equal to the *direct sum* $r(\eta) \oplus c$. If there exists a complement of the endomorphism η, then η is said to be a *splitting endomorphism* of $p/0$ or to *split* p; and if η induces a splitting endomorphism of $x/0$ for every η-admissible x, η is a *uniform splitting endomorphism* of $p/0$.

Throughout the remainer of these statements from [4] η denotes an endomorphism of $p/0$ for an arbitrary element p in P.

The element x is said to be *η-automorphic* if $x\eta \leq x \leq p$ and if $y\eta \leq y \leq x$ implies $y\eta = y$.

THEOREM B. *If the endomorphism η of $p/0$ splits p uniformly, then the complement of η is uniquely determined, η-automorphic, and equal to the sum of all η-automorphic elements.*

COROLLARY B. *If η splits p uniformly with complement $c(\eta)$ and if x is an η-admissible element, then $xc(\eta)$ is the uniquely determined complement of the endomorphism which η induces in $x/0$.*

THEOREM C. *If η is an endomorphism of $p/0$ and if for every η-admissible x there exists a positive integer $h = h(x)$ such that $x\eta^h = x\eta^{h+1} = \cdots$ and such that $k(\eta^h) = k(\eta^{h+1}) = \cdots = r(\eta)$, then $x = xr(\eta) \bigoplus x\eta^h$, $x\eta^h$ is a complement of the endomorphism induced in $x/0$, and η is uniformly splitting.*

2. Direct decompositions and decomposition endomorphisms. The element p is the *direct sum of a and b* ($p = a \bigoplus b$) if $a + b = p$ and $ab = 0$. (Throughout the paper p represents an arbitrary element in P.)

If $p = a \bigoplus b$ and if $x \leq p$, there exist uniquely determined mappings α and β which we define by the following equations:

$$x\alpha = (b + x)a \quad \text{and} \quad x\beta = (a + x)b.$$

It follows from these equations that, if $y \leq a$, $y\alpha = y$ and $y\beta = 0$. Similarly, if $z \leq b$, $z\alpha = 0$ and $z\beta = z$.

Upon the basis of these definitions and Dedekind's modular law it can be shown without difficulty that the following properties are possessed by α and β:

1) α and β are endomorphisms of $p/0$.

2) $\alpha^2 = \alpha$ and $\beta^2 = \beta$.

3) $\alpha\beta = \beta\alpha = 0$.

4) $k(\alpha) = p\beta = b$, $k(\beta) = p\alpha = a$, and thus
$$p = a \bigoplus b = p\alpha \bigoplus p\beta = k(\beta) \bigoplus k(\alpha).$$

The pair of endomorphisms α, β is termed a *pair of complementary decomposition endomorphisms.* Either member of the pair is called a decomposition endomorphism. Constant use is made of these facts on decomposition endomorphisms without reference. Likewise, whenever α and β are given as decomposition endomorphisms, the notations just used are to be understood. Similarly, for pairs of decomposition endomorphisms δ, ϵ and ρ, σ the following notations are used:

$$p = d \oplus e = r \oplus s$$

$$d = p\delta = k(\epsilon) \qquad e = p\epsilon = k(\delta)$$

$$r = p\rho = k(\sigma) \qquad s = p\sigma = k(\rho)$$

and for $x \leq p$

$$x\delta = (e + x)d \qquad x\epsilon = (d + x)e$$

$$x\rho = (s + x)r \qquad x\sigma = (r + x)s.$$

The element p is the *direct sum of* $s(1), s(2), \cdots, s(n)$ $(p = s(1) \oplus s(2) \oplus \cdots \oplus s(n))$ if, for every i, p is the direct sum of $s(i)$ and of the sum of the $s(j)$ for $j \neq i$. The following property of direct sums (See, for example, [1], p. 66.) is useful:

$$s(1) \oplus s(2) \oplus \cdots \oplus s(n) = [s(1) \oplus \cdots \oplus s(i)] \oplus [s(i+1) \oplus \cdots \oplus s(n)].$$

3. Preparatory lemmas. In preparation for the proof of the refinement theorems, we deduce the following lemmas.

LEMMA 1. *If* $p = a \oplus b$ *and if* $a \leq s \leq p$, *then* $s = a \oplus sb$.

Proof. $a(sb) = (ab)s = 0s = 0$, and $s = s(a + b) = a + sb$, by Dedekind's law.

LEMMA 2. *If* α, β *and* ρ, σ *are pairs of complementary decomposition endomorphisms of* $p/0$ *and if* $\alpha\rho\alpha$ *induces an automorphism of* $p\alpha/0$, *then* $p = p\alpha\rho \oplus p\beta$.

Proof. From the definitions of the decomposition endomorphisms, it follows that $a\alpha\rho\alpha = [b + (a + s)r]a$. Since $\alpha\rho\alpha$ induces an automorphism of $p\alpha/0 = a/0$, $a = a\alpha\rho\alpha = a\rho\alpha$. Thus $a = [b + (a + s)r]a$, and $a \leq b + (a + s)r$. Hence $p = a + b \leq (a + s)r + b \leq p$, and thus

$$p = (a + s)r + b = a\rho + p\beta = p\alpha\rho + p\beta.$$

Moreover, $(p\alpha\rho)(p\beta) \leq p\alpha\rho$ and ρ is an endomorphism of $p/0$, mapping $p\alpha/0$ $(p\alpha \leq p)$ upon $p\alpha\rho/0$ $(p\alpha\rho \leq p\rho)$. Hence there exists an element $x \leq p\alpha$ such that $x = x\alpha$ and $x\rho (= x\alpha\rho) = (p\alpha\rho)(p\beta)$. Since $(p\alpha\rho)(p\beta) \leq p\beta$,

$$0 = [(p\alpha\rho)(p\beta)]\alpha = x\alpha\rho\alpha.$$

Since $\alpha\rho\alpha$ induces an automorphism of $p\alpha/0$ for $p\alpha \geq x$, the equality $x\alpha\rho\alpha = 0$ implies that $x = 0$. Hence $0 = x\rho = (p\alpha\rho)(p\beta)$, and $p = p\alpha\rho \oplus p\beta$.

LEMMA 3. *If* α, β *is a pair of complementary decomposition endo-*

morphisms of $p/0$ and if the decomposition endomorphism ρ of $p/0$ satisfies the conditions $p\rho \leq p\alpha$ and $p\beta \leq k(\rho)$, then $\rho = \rho\alpha = \alpha\rho$.

Proof. Since $x\rho \leq p\rho \leq p\alpha$ for any $x \leq p$, $(x\rho)\alpha = x\rho$, or $\rho = \rho\alpha$.

It is to be shown next that $k(\alpha\rho) = k(\rho)$. Since α, β and ρ are decomposition endomorphisms, $p = a \bigoplus b = p\alpha \bigoplus p\beta = r \bigoplus s = p\rho \bigoplus s$. By hypothesis, $p\beta = b \leq k(\rho) = s$, and hence $s\alpha\rho = [s + (b + s)a]r = [s + sa]r = sr = 0$. Thus $k(\rho) \leq k(\alpha\rho)$. Moreover, $b = k(\alpha) \leq k(\alpha\rho)$. Hence

$$\begin{aligned} k(\alpha\rho) &= k(\alpha\rho)[a + b] \\ &= b + ak(\alpha\rho) \quad \text{(by Dedekind's law).} \end{aligned}$$

Since $0 = [ak(\alpha\rho)]\alpha\rho = [ak(\alpha\rho)]\rho$, it follows that $ak(\alpha\rho) \leq k(\rho)$. Thus, by hypothesis, $k(\alpha\rho) = b + ak(\alpha\rho) \leq k(\rho)$. Hence $k(\alpha\rho) = k(\rho)$.

Since $\alpha^2 = \alpha$, for any $x \leq p$, $x\alpha\rho = (x\alpha)\alpha\rho$. It follows then from the fact that the product $\alpha\rho$ of the endomorphisms α and ρ of $p/0$ is an endomorphism of $p/0$ that $x + k(\alpha\rho) = x\alpha + k(\alpha\rho)$. Hence, by the preceding paragraph, $x + k(\rho) = x\alpha + k(\rho)$. As a result of applying ρ to this equality, $x\rho = [x + k(\rho)]\rho = [x\alpha + k(\rho)]\rho = x\alpha\rho$, or $\rho = \alpha\rho$, as required.

COROLLARY 1. *If α, β and δ, ϵ are pairs of complementary decomposition endomorphisms of $p/0$, if $p\alpha = u \bigoplus v$, and if $\alpha\delta\alpha$ induces an automorphism of $u/0$, then $p = u\delta \bigoplus v \bigoplus p\beta$.*

Proof. Since $p = p\alpha \bigoplus p\beta = u \bigoplus (v \bigoplus p\beta)$, there exist complementary decomposition endomorphisms ρ and σ such that $p\rho = u$ and $p\sigma = v \bigoplus p\beta$. Furthermore, $p\rho = u \leq p\alpha$ and $p\beta \leq v + p\beta = k(\rho)$. Hence, by Lemma 3, $\rho = \rho\alpha = \alpha\rho$. If $x \leq u$, $x\alpha\delta\alpha \leq u\alpha\delta\alpha = u$ and thus $x\rho = x$ and $(x\alpha\delta\alpha)\rho = x\alpha\delta\alpha$. Hence $x\alpha\delta\alpha = x\rho\alpha\delta\alpha\rho = x\rho\delta\rho$, and thus $\alpha\delta\alpha$ and $\rho\delta\rho$ induce the same automorphism of $u/0$. Hence by Lemma 2, $p = p\rho\delta \bigoplus p\sigma = u\delta \bigoplus v \bigoplus p\beta$.

LEMMA 4. *If α, β and δ, ϵ are pairs of complementary decomposition endomorphisms of $p/0$, then*

1) $(\alpha\delta\alpha)(\alpha\epsilon\alpha) = (\alpha\epsilon\alpha)(\alpha\delta\alpha)$, $\delta\alpha\epsilon = \delta\beta\epsilon$, *and* $\alpha\delta\beta = \alpha\epsilon\beta$.

and for $x\alpha\epsilon\alpha \leq x \leq p\alpha$

2) $x = x\alpha\delta\alpha + x\alpha\epsilon\alpha$.

3) $xr(\alpha\delta\alpha\epsilon\alpha) = xr(\alpha\delta\alpha) \bigoplus xr(\alpha\epsilon\alpha)$.

4) *An automorphism is induced in $xr(\alpha\delta\alpha)/0$ by $\alpha\epsilon\alpha$ and in $xr(\alpha\epsilon\alpha)/0$ by $\alpha\delta\alpha$.*

Proof. The proofs of 1) and (with the use of 2)) 3) follow directly from the proofs of lemmas 1 to 8 of [6] by a change of notation.

Proof of 2).

$$b + a\{b + [(x + d)e]\} + a\{b + [(x + e)d]\}$$
$$= \{b + [(x + d)e]\}\{a + b\} + \{b + [(x + e)d]\}\{a + b\}$$
(by Dedekind's law)
$$= b + (x + d)e + (x + e)d$$
$$= b + (x + d)[e + \{(x + e)d\}] \quad \text{(by Dedekind's law and } (x + e)d \leq x + d\text{)}$$
$$= b + (x + d)[(x + e)(e + d)] \quad \text{(by Dedekind's law and } x + e \geq e\text{)}$$
$$= b + (x + d)(x + e)$$
$$= b + x + (x + d)e \quad \text{(by Dedekind's law and } x + d \geq x\text{)}$$
$$= [b + x + (x + d)e][a + b]$$
$$= b + a[b + x + (x + d)e] \quad \text{(by Dedekind's law)}$$
$$= b + x + a[b + (x + d)e] \quad \text{(by Dedekind's law and } x \leq a\text{)}$$
$$= b + x \quad \text{(by hypothesis).}$$

Hence $a[b + x\alpha\epsilon\alpha + x\alpha\delta\alpha] = a[b + x]$. From this equality, Dedekind's law, $x \leq a$, and $x\alpha\epsilon\alpha + x\alpha\delta\alpha \leq a$, it follows that $x\alpha\epsilon\alpha + x\alpha\delta\alpha = x$.

Proof of 4). Since $\{xk[(\alpha\delta\alpha)^i]\}(\alpha\epsilon\alpha)^i = 0$,

$$\{xk[(\alpha\delta\alpha)^i]\}(\alpha\epsilon\alpha)(\alpha\delta\alpha)^i = \{xk[(\alpha\delta\alpha)^i]\}(\alpha\delta\alpha)^i(\alpha\epsilon\alpha) = 0,$$

by 1), and hence $\{xk[(\alpha\delta\alpha)^i]\}(\alpha\epsilon\alpha) \leq k[(\alpha\delta\alpha)^i]$. Also $\{xk[(\alpha\delta\alpha)^i]\}(\alpha\epsilon\alpha) \leq x\alpha\epsilon\alpha \leq x$, and thus, for all i,

$$\{xk[(\alpha\delta\alpha)^i]\}(\alpha\epsilon\alpha) \leq xk[(\alpha\delta\alpha)^i] \leq xr(\alpha\delta\alpha).$$

Hence

$$(1) \qquad \sum_{i=1}^{\infty}[\{xk[(\alpha\delta\alpha)^i]\}\alpha\epsilon\alpha] = [\sum_{i=1}^{\infty}\{xk[(\alpha\delta\alpha)^i]\}]\alpha\epsilon\alpha$$
$$= [x\sum_{i=1}^{\infty}k[(\alpha\delta\alpha)^i]]\alpha\epsilon\alpha \quad \text{(by Postulate (*))}$$
$$= [xr(\alpha\delta\alpha)]\alpha\epsilon\alpha \leq xr(\alpha\delta\alpha).$$

It is clear that $\{xk[(\alpha\delta\alpha)^i]\}(\alpha\delta\alpha)^i = 0 \leq [xr(\alpha\delta\alpha)]\alpha\epsilon\alpha$. As induction hypothesis, assume that $\{xk[(\alpha\delta\alpha)^i]\}(\alpha\delta\alpha)^j \leq [xr(\alpha\delta\alpha)]\alpha\epsilon\alpha$ for $0 < j \leq i$. Let $t = \{xk[(\alpha\delta\alpha)^i]\}$. Then, since $t \leq t\alpha\delta\alpha + t\alpha\epsilon\alpha$,

$$t(\alpha\delta\alpha)^{j-1} \leq t(\alpha\delta\alpha)^j + t(\alpha\epsilon\alpha)(\alpha\delta\alpha)^{j-1} = t(\alpha\delta\alpha)^j + t(\alpha\delta\alpha)^{j-1}(\alpha\epsilon\alpha).$$

By the induction hypothesis, $t(\alpha\delta\alpha)^j \leq [xr(\alpha\delta\alpha)]\alpha\epsilon\alpha$. Since $t \leq k[(\alpha\delta\alpha)^i]$,

$$t(\alpha\delta\alpha)^{j-1} \leq k[(\alpha\delta\alpha)^i] \leq r(\alpha\delta\alpha),$$

and

$$t(\alpha\delta\alpha)^{j-1} \leq x(\alpha\delta\alpha)^{j-1} \leq x \qquad \text{(by 2))}.$$

Hence $[t(\alpha\delta\alpha)^{j-1}]\alpha\epsilon\alpha \leq [xr(\alpha\delta\alpha)]\alpha\epsilon\alpha$, and thus $t(\alpha\delta\alpha)^{j-1} \leq [xr(\alpha\delta\alpha)]\alpha\epsilon\alpha$. Hence by complete induction $t = xk[(\alpha\delta\alpha)^i] \leq [xr(\alpha\delta\alpha)]\alpha\epsilon\alpha$. This statement is true for all i, and hence

$$\sum_{i=1}^{\infty} xk[(\alpha\delta\alpha)^i] = xr(\alpha\delta\alpha) \leq [xr(\alpha\delta\alpha)]\alpha\epsilon\alpha.$$

This fact together with (1) shows that $[xr(\alpha\delta\alpha)]\alpha\epsilon\alpha = xr(\alpha\delta\alpha)$.

Moreover, $k(\alpha\epsilon\alpha)xr(\alpha\delta\alpha) \leq xr(\alpha\epsilon\alpha)xr(\alpha\delta\alpha) = 0$, by 3). Hence $\alpha\epsilon\alpha$ induces an automorphism of $xr(\alpha\delta\alpha)/0$. That $\alpha\delta\alpha$ induces an automorphism of $xr(\alpha\epsilon\alpha)/0$ can be proved in a like manner.

LEMMA 5. *If α,β and δ,ϵ are pairs of complementary decomposition endomorphisms of $p/0$ and if $\alpha\delta\alpha\epsilon\alpha$ is a uniformly splitting endomorphism of $p/0$ with complement c, then there exists a direct decomposition $p\alpha = u \oplus v$, where $u = c \oplus p\alpha r(\alpha\delta\alpha)$ and $v = p\alpha r(\alpha\epsilon\alpha)$, and this direct decomposition is uniquely determined by the properties*:

1) $v \leq r(\alpha\epsilon\alpha)$ *and*

2) *an automorphism is induced in $u/0$ by $\alpha\epsilon\alpha$ and in $v/0$ by $\alpha\delta\alpha$.*

Proof. It is to be shown first that $u = c \oplus p\alpha r(\alpha\delta\alpha)$ and $v = p\alpha r(\alpha\epsilon\alpha)$ constitute a direct decomposition of $p\alpha = a$, satisfying 1) and 2). Since $k(\alpha) \leq r(\alpha\delta\alpha\epsilon\alpha)$ and since $p = a \oplus k(\alpha)$, it is a consequence of Lemma 1 that $r(\alpha\delta\alpha\epsilon\alpha) = k(\alpha) \oplus ar(\alpha\delta\alpha\epsilon\alpha)$. Since $a\alpha\epsilon\alpha \leq a$, it follows from Lemma 4 that $ar(\alpha\delta\alpha\epsilon\alpha) = ar(\alpha\delta\alpha) \oplus ar(\alpha\epsilon\alpha)$. Thus

$$r(\alpha\delta\alpha\epsilon\alpha) = k(\alpha) \oplus ar(\alpha\delta\alpha) \oplus ar(\alpha\epsilon\alpha).$$

Because of the hypothesis of splitting, $p = r(\alpha\delta\alpha\epsilon\alpha) \oplus c$, and hence

$$p = k(\alpha) \oplus ar(\alpha\delta\alpha) \oplus ar(\alpha\epsilon\alpha) \oplus c.$$

Since $c = c\alpha\delta\alpha\epsilon\alpha \leq a$, the last three summands and hence their sum are part of a. Lemma 1 and $ak(\alpha) = 0$ then imply that

$$p\alpha = c \oplus ar(\alpha\delta\alpha) \oplus ar(\alpha\epsilon\alpha) = u \oplus v.$$

Since $v = ar(\alpha\epsilon\alpha) \leq r(\alpha\epsilon\alpha)$, 1) is satisfied. By Lemma 4, an auto-

morphism is induced in $ar(\alpha\epsilon\alpha)/0 = v/0$ by $\alpha\delta\alpha$. The next step is to prove that $\alpha\epsilon\alpha$ induces an automorphism of $u/0$.

In this regard it is to be shown first that $c\alpha\epsilon\alpha r(\alpha\delta\alpha\epsilon\alpha) = 0$. Let $c\alpha\epsilon\alpha r(\alpha\delta\alpha\epsilon\alpha) = t$. Since $\alpha\epsilon\alpha$ maps $c/0$ upon $c\alpha\epsilon\alpha/0$ and since $t \leq c\alpha\epsilon\alpha$, there exists $s \leq c$ such that $s\alpha\epsilon\alpha = t$. Thus $s\alpha\delta\alpha\epsilon\alpha = s\alpha\epsilon\alpha\delta\alpha$ (by 1) of Lemma 4) $= t\alpha\delta\alpha \leq [ar(\alpha\delta\alpha\epsilon\alpha)]\alpha\delta\alpha$. By Lemma 4,

$$[ar(\alpha\delta\alpha\epsilon\alpha)]\alpha\delta\alpha = [ar(\alpha\delta\alpha) \bigoplus ar(\alpha\epsilon\alpha)]\alpha\delta\alpha.$$

Moreover, $[ar(\alpha\delta\alpha)]\alpha\delta\alpha \leq ar(\alpha\delta\alpha)$ and, by Lemma 4, $[ar(\alpha\epsilon\alpha)]\alpha\delta\alpha = ar(\alpha\epsilon\alpha)$. Thus

$$[ar(\alpha\delta\alpha\epsilon\alpha)]\alpha\delta\alpha \leq ar(\alpha\delta\alpha\epsilon\alpha) \leq r(\alpha\delta\alpha\epsilon\alpha).$$

Hence, by 1) of Theorem A, $s \leq r(\alpha\delta\alpha\epsilon\alpha)$. Consequently, since $s \leq c$, $s \leq cr(\alpha\delta\alpha\epsilon\alpha) = 0$. Hence $s\alpha\epsilon\alpha = t = c\alpha\epsilon\alpha r(\alpha\delta\alpha\epsilon\alpha) = 0$. Similarly,

$$c\alpha\delta\alpha r(\alpha\delta\alpha\epsilon\alpha) = 0.$$

Since c is a complement of $\alpha\delta\alpha\epsilon\alpha$,

$$c(\alpha\epsilon\alpha)(\alpha\delta\alpha\epsilon\alpha) = c(\alpha\epsilon\alpha)(\alpha\delta\alpha)(\alpha\epsilon\alpha) = c(\alpha\delta\alpha\epsilon\alpha)(\alpha\epsilon\alpha) = c\alpha\epsilon\alpha.$$

Thus, $c\alpha\epsilon\alpha$ is $\alpha\delta\alpha\epsilon\alpha$-admissible, and since $\alpha\delta\alpha\epsilon\alpha$ is uniformly splitting,

$$c\alpha\epsilon\alpha = c(c\alpha\epsilon\alpha) \bigoplus c\alpha\epsilon\alpha r(\alpha\delta\alpha\epsilon\alpha) = c(c\alpha\epsilon\alpha),$$

by Corollary B. Hence $c\alpha\epsilon\alpha \leq c$ and $c = (c\alpha\epsilon\alpha)\alpha\delta\alpha \leq c\alpha\delta\alpha$. Similarly $c\alpha\delta\alpha \leq c$ and $c \leq c\alpha\epsilon\alpha$. Hence $c = c\alpha\epsilon\alpha$ (and $c = c\alpha\delta\alpha$). Since, by Lemma 4, $\alpha\epsilon\alpha$ induces an automorphism of $ar(\alpha\delta\alpha)/0$, $[ar(\alpha\delta\alpha)]\alpha\epsilon\alpha = ar(\alpha\delta\alpha)$. Hence

$$u\alpha\epsilon\alpha = [c + ar(\alpha\delta\alpha)]\alpha\epsilon\alpha = c + ar(\alpha\delta\alpha) = u.$$

From $uk(\alpha\epsilon\alpha) \leq ar(\alpha\epsilon\alpha) = v$ it follows that $uk(\alpha\epsilon\alpha) \leq uv = 0$. Thus $\alpha\epsilon\alpha$ induces an automorphism of $u/0$, and the proof of the existence is completed.

Consider a direct decomposition $p\alpha = u' \bigoplus v'$ satisfying conditions 1) and 2). By 1), $v' \leq ar(\alpha\epsilon\alpha) = v$. Hence, by Lemma 1,

$$v = v' \bigoplus vu'. \tag{2}$$

By 2), $\alpha\epsilon\alpha$ induces an automorphism of $u'/0$. Thus $(\alpha\epsilon\alpha)^i$ induces an automorphism of $u'/0$, and $u'k[(\alpha\epsilon\alpha)^i] = 0$ for all i. Hence $u'r(\alpha\epsilon\alpha) = 0$ and thus $u'v = 0$. From (2) it now follows that $v = v'$.

Since $v = v'$, $a = u \bigoplus ar(\alpha\epsilon\alpha) = u' \bigoplus ar(\alpha\epsilon\alpha)$, and, by 2), $u\alpha\epsilon\alpha = u$ and $u'\alpha\epsilon\alpha = u'$. Hence u and u' are both complements of the endomorphism which $\alpha\epsilon\alpha$ induces in $a/0$ $(a \geq a\alpha\epsilon\alpha)$. Consider any x such that $x \leq a$ and

$x\alpha\epsilon\alpha \leq x$. By 2) of Lemma 4, $x\alpha\delta\alpha \leq x$. Hence $x\alpha\delta\alpha\epsilon\alpha = (x\alpha\delta\alpha)\alpha\epsilon\alpha \leq x\alpha\epsilon\alpha \leq x$. Thus, since $\alpha\delta\alpha\epsilon\alpha$ is uniformly splitting, $x = xc \oplus xr(\alpha\delta\alpha\epsilon\alpha)$, by Corollary B. By 3) of Lemma 4, $xr(\alpha\delta\alpha\epsilon\alpha) = xr(\alpha\delta\alpha) \oplus xr(\alpha\epsilon\alpha)$, and hence $x = [xc \oplus xr(\alpha\delta\alpha)] \oplus xr(\alpha\epsilon\alpha)$.

It is known that $(xc)\alpha\delta\alpha \leq x\alpha\delta\alpha c\alpha\delta\alpha \leq xc$ and $(xc)\alpha\epsilon\alpha \leq x\alpha\epsilon\alpha c\alpha\epsilon\alpha \leq xc$. Moreover, since xc is a complement,

$$xc = (xc)\alpha\delta\alpha\epsilon\alpha = [(xc)\alpha\delta\alpha]\alpha\epsilon\alpha \leq (xc)\alpha\epsilon\alpha.$$

Thus $(xc)\alpha\epsilon\alpha = xc$. By 4) of Lemma 4, $[xr(\alpha\delta\alpha)]\alpha\epsilon\alpha = xr(\alpha\delta\alpha)$, and thus

$$[xc \oplus xr(\alpha\delta\alpha)]\alpha\epsilon\alpha = xc \oplus xr(\alpha\delta\alpha).$$

Hence $\alpha\epsilon\alpha$ splits x. Thus the endomorphism which $\alpha\epsilon\alpha$ induces in $a/0$ splits a unformily. Hence, by Theorem B, its complement is unique, and $u' = u$. Thus the uniqueness of the direct decomposition of $p\alpha$ has been shown.

LEMMA 6. *If α, β and δ, ϵ are pairs of complementary decomposition endomorphisms of $p/0$ and if the direct decompositions*

$$p\alpha = a' \oplus a'', \quad p\beta = b' \oplus b'', \quad p\delta = d' \oplus d'', \text{ and } p\epsilon = e' \oplus e''$$

have the properties that

α induces an isomorphism of $d'/0$ upon $a'/0$ and of $e''/0$ upon $a''/0$,
β induces an isomorphism of $e'/0$ upon $b'/0$ and of $d''/0$ upon $b''/0$,
δ induces an isomorphism of $a'/0$ upon $d'/0$ and of $b''/0$ upon $d''/0$, and
ϵ induces an isomorphism of $b'/0$ upon $e'/0$ and of $a''/0$ upon $e''/0$

then the following equations are satisfied:

$$\begin{aligned} p\alpha \oplus b' &= p\alpha \oplus e' & \qquad p\alpha \oplus b'' &= p\alpha \oplus d'' \\ p\beta \oplus a' &= p\beta \oplus d' & \qquad p\beta \oplus a'' &= p\beta \oplus e'' \\ p\delta \oplus e' &= p\delta \oplus b' & \qquad p\delta \oplus e'' &= p\delta \oplus a'' \\ p\epsilon \oplus d' &= p\epsilon \oplus a' & \qquad p\epsilon \oplus d'' &= p\epsilon \oplus b''. \end{aligned}$$

Proof. Since $\beta\epsilon\beta$ induces an automorphism of $b'/0$,

$$p = b'\epsilon \oplus b'' \oplus p\alpha = e' \oplus b'' \oplus p\alpha,$$

by Corollary 1. Furthermore, $b' = e'\beta = (e' + a)b \leq e' + a = p\alpha \oplus e'$, and thus $p\alpha \oplus b' \leq p\alpha \oplus e'$.

Since $\beta\epsilon\beta$ induces an automorphism of $b''/0$, it follows from the same

corollary that $p = p\alpha \oplus b' \oplus b''\delta = p\alpha \oplus b' \oplus d''$. Hence, by Lemma 1 and the fact that $p\alpha \oplus b' \leq p\alpha \oplus e'$,

$$p\alpha \oplus e' = (p\alpha \oplus b') \oplus d''(p\alpha \oplus e'). \tag{3}$$

By Dedekind's law, $(e' + d'')(a + e') = e' + d''(a + e')$ and also $(e' + d'')(a + e') = e' + a(c' + d'')$. Thus $e' + d''(a + e') = e' + a(e' + d'')$, and hence $[e' + d''(a + e')]\beta = [e' + a(e' + d'')]\beta$. Thus $b' + [d''(a + e')]\beta = b'$, and $[d''(a + e')]\beta \leq b'$. Furthermore, $d'(a + e') \leq d''$, and thus $[d''(a + e')]\beta \leq d''\beta = b''$ (by hypothesis). Hence $[d''(a + e')]\beta \leq b'b'' = 0$. Thus $d''(a + e') \leq k(\beta) = p\alpha$, and hence $d''(a + e') \leq d''p\alpha = 0$. Hence, by (3), $p\alpha \oplus b' = p\alpha \oplus e'$. The other equations of the assertion can be proved in a like manner.

4. Proof of the refinement theorems. To prove the Special Refinement Theorem let α, β and δ, ϵ denote the pairs of complementary decomposition endomorphisms such that $p\alpha = a$, $p\beta = b$, $p\delta = d$, and $p\epsilon = e$. Lemma 5 asserts the existence of uniquely determined elements a', a'', d', and d'' such that

$$a = a' \oplus a'', \quad d = d' \oplus d'', \quad a'' \leq r(\alpha\delta\alpha), \quad d'' \leq r(\delta\alpha\delta),$$

and an automorphism is induced in $a'/0$ by $\alpha\delta\alpha$, in $a''/0$ by $\alpha\epsilon\alpha$, in $d'/0$ by $\delta\alpha\delta$, and in $d''/0$ by $\delta\beta\delta$.

Let the following definitions hold:

$$b' = b(d' \oplus e), \quad b'' = d''\beta, \quad e' = e(a' \oplus b), \quad \text{and} \quad e'' = a''\epsilon.$$

As in [1] (p. 78) it can be shown that

$$b = b' \oplus b'' \quad \text{and} \quad e = e' \oplus e''.$$

It is to be shown next that these decompositions of a, b, d, and e satisfy the hypotheses of Lemma 6. The following facts can be deduced by essentially the same arguments as in [1] (p. 78):

a) α induces an isomorphism of $e''/0$ upon $a''/0$.

b) β induces an isomorphism of $d''/0$ upon $b''/0$.

c) δ induces an isomorphism of $b''/0$ upon $d''/0$.

d) ϵ induces an isomorphism of $a''/0$ upon $e''/0$.

e) $d = a'\delta \oplus d(a'' \oplus b)$.

f) $\delta\alpha\delta$ induces an automorphism of $a'\delta/0$.

It is to be shown next that $[ar(\alpha\delta\alpha)]\delta \leq r(\delta\alpha\delta)$. Since $\{ak[(\alpha\delta\alpha)^i]\}(\alpha\delta\alpha)^i = 0$, $[\{ak(\alpha\delta\alpha)^i]\}\alpha\delta(\delta\alpha\delta)^i = 0$ and thus

$$\{ak[(\alpha\delta\alpha)^i]\}\alpha\delta = \{ak[(\alpha\delta\alpha)^i]\}\delta \leq k[(\delta\alpha\delta)^i] \text{ for all } i.$$

Hence $\sum_{i=1}^{\infty}\{ak[(\alpha\delta\alpha)^i]\}\delta \leq \sum_{i=1}^{\infty} k[(\delta\alpha\delta)^i]$. Thus, by Postulate (*) and the fact that δ is additive,

$$[ar(\alpha\delta\alpha)]\delta \leq r(\delta\alpha\delta). \tag{4}$$

Moreover,

$$[d(a'' \oplus b)]\delta\alpha = [d(a'' \oplus b)]\alpha \leq (a'' \oplus b)\alpha = a''. \tag{5}$$

Since $a'' \leq ar(\alpha\delta\alpha)$, $a''\delta \leq [ar(\alpha\delta\alpha)]\delta \leq r(\delta\alpha\delta)$, by (4). Hence by (5), $[d(a'' \oplus b)]\delta\alpha\delta \leq a''\delta \leq r(\delta\alpha\delta)$. Thus, by 1) of Theorem A, $d(a'' \oplus b) \leq r(\delta\alpha\delta)$.

From $d = a'\delta \oplus d(a'' \oplus b)$, $d(a'' \oplus b) \leq dr(\delta\alpha\delta)$, and Lemma 1, it follows that $dr(\delta\alpha\delta) = d(a'' \oplus b) \oplus dr(\delta\alpha\delta)a'\delta$. Since $\delta\alpha\delta$, and thus $(\delta\alpha\delta)^i$, induces an automorphism of $a'\delta/0$, $a'\delta k[(\delta\alpha\delta)^i] = 0$, for all i. Hence $\sum_{i=1}^{\infty} a'\delta k[(\delta\alpha\delta)^i] = a'\delta r(\delta\alpha\delta) = 0$. Thus $dr(\delta\alpha\delta) = d(a'' \oplus b)$. By 4) of Lemma 4, $\delta\beta\delta$ induces an automorphism of $dr(\delta\alpha\delta)/0$, and hence

an automorphism is induced in $d(a'' \oplus b)/0$ by $\delta\beta\delta$.

The decomposition e) of d thus meets the requirements which, by Lemma 5, determine uniquely the decomposition $d = d' \oplus d''$. Hence

$$d' = a'\delta \quad \text{and} \quad d'' = d(a'' \oplus b). \tag{6}$$

Likewise it is seen that

$$a' = d'\alpha \quad \text{and} \quad a'' = a(d'' \oplus e). \tag{7}$$

It is a consequence of (6), (7), and the facts that $\delta\alpha\delta$ induces an automorphism of $d'/0$ and that $\alpha\delta\alpha$ induces an automorphism of $a'/0$ that δ induces an isomorphism of $a'/0$ upon $d'/0$ and α induces an isomorphism of $d'/0$ upon $a'/0$.

Moreover, $k(\beta)e' = ae' = a[e(a' \oplus b)] = ea'$ (by Dedekind's law). Furthermore,

$$(ea')\alpha\delta\alpha = (ea')\delta\alpha \leq (e\delta)\alpha = 0, \tag{8}$$

and since $\alpha\delta\alpha$ induces an automorphism of $a'/0$, the equality (8) implies that

$ea' = 0$. Thus $e'k(\beta) = 0$. Similarly, since $\delta\alpha\delta$ induces an automorphism of $d'/0$, $b'k(\epsilon) = 0$.

Since $d' = a'\delta$, $d'\delta = a'\delta^2 = a'\delta$. Thus, since δ is an endomorphism of $p/0$, $d' + e = a' + e$. Hence $b' = b(d' + e) = b(a' + e)$. Moreover, by the modular law,

$$(a' + e)(a' + b) = a' + b(a' + e) = a' + b'$$

and

$$(a' + e)(a' + b) = a' + e(a' + b) = a' + e'.$$

Thus $(a' + b')\beta = (a' + e')\beta$, and hence $b' = b'\beta = e'\beta$. Since $a' = d'\alpha$, it follows in a similar manner that $b'\epsilon = e'$. Consequently β induces an isomorphism of $e'/0$ upon $b'/0$, and ϵ induces an isomorphism of $b'/0$ upon $e'/0$.

The proof that the decompositions of a, b, d, and e satisfy the hypotheses of Lemma 6 is thus completed. Hence the decompositions satisfy the equations of the conclusion of that lemma, and we use this fact (as in [1] (pp. 79-80) without repeating the arguments here) to complete the proof of the theorem.

With the preceding results now available, the General Refinement Theorem can be deduced from the Special Refinement Theorem by means of a double induction carried out precisely as in [1] (pp. 81-84), where only structural arguments are used. The following two additional lemmas are required to execute this proof:

LEMMA 7. *If* $p = d \oplus a(1) \oplus \cdots \oplus a(n) = d \oplus b$, *then*

1) $d \oplus a(i) = d \oplus b(i)$, *where* $b(i) = [d \oplus a(i)]b$, *and*

2) $b = b(1) \oplus \cdots \oplus b(n)$.

LEMMA 8. *If the Splitting Hypothesis is satisfied by* p, *then it is satisfied by every direct summand of* p.

The proofs of the analogous lemmas in [1] (p. 70 and p. 72) can be used to derive Lemma 7 and, with a simple modification, Lemma 8. The proof of the theorem is not reproduced in our notation.

5. Relationship to theorems of Ore, Kurosch, and Grayev. Previous investigations relating to direct decompositions in Dedekind structures have been made in [3], [5], [6], and [7].

By comparison with [7], it is seen that what are there termed "reductions" are products of decomposition endomorphisms such as $\alpha\delta$ (in the previous notation) and that the "null-factors" are kernels of powers of products of decomposition endomorphisms. Moreover, the condition of the existence of "regular decompositions" can be stated as follows in terms of decomposition endomorphisms: If α and δ are decomposition endomorphisms such that $x\alpha\delta \leq x$, then $x = x(\alpha\delta)^n \oplus xk[(\alpha\delta)^m]$. This splitting condition is much stronger than our splitting hypothesis.

The main theorem of [7] (the assertion of which is that in two different direct decompositions into direct indecomposable summands both sides contain the same number of summands directly similar in pairs) is proved under the assumption of the existence of regular decompositions. It is to be noted here that we restrict ourselves to complete lattices, while Ore's theorem is general. By means of Lemma 4, Theorem C, and a series of computations (here omitted) it can be shown that the Splitting Hypothesis is a consequence of Ore's condition of the existence of regular decompositions. Thus for complete lattices the following obvious consequence of our theorem is easily seen to contain Ore's results:

If the condition of the existence of regular decompositions is satisfied, then two different direct decompositions into direct indecomposable summands are exchange isomorphic.

The "center" in connection with a pair of direct decompositions has been used by Kurosch ([5], p. 190) and by Baer ([2], p. 524). The refinement theorems of Kurosch and Grayev (Theorem 1 of [6] and Theorem 1 of [3]) are proved under the assumption that the sequence of centers of a pair of direct decompositions attains zero at a finite step, while we have proved the refinement theorems by assuming the Splitting Hypothesis for every pair of complementary decomposition endomorphisms. It is of interest to note that if the n-th center of a pair of decompositions is zero for an integer n, $(\rho\beta\rho\alpha\rho)^{n+1} = 0$ for α, β a pair of complementary decomposition endomorphisms and ρ a decomposition endomorphism. By comparing this condition with the Splitting Hypothesis, it is noted that $\rho\beta\rho\alpha\rho$ may be uniformly splitting and $(\rho\beta\rho\alpha\rho)^{n+1} \neq 0$.

In regard to the relationship between the two results, it can be stated that the Splitting Hypothesis is a weaker hypothesis on all pairs of complementary decomposition endomorphisms, while Kurosch's hypothesis is a stronger condition on a single pair of direct decompositions. Moreover, the conclusion of direct similarity of refinements is weaker than that of exchange

isomorphy. Thus for all pairs of decomposition endomorphisms our result is stronger than that of Kurosch. However, since our theorem is stated from the "universal" point of view, it is independent of the "individualistic" theorem of Kurosch.

TEMPLE UNIVERSITY.

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A GENERALIZATION OF A THEOREM OF JACOBSON.*

By I. N. HERSTEIN.

In [2], Theorem 11, Jacobson proved a theorem which at the same time generalizes the theorem of Wedderburn that every finite division ring is commutative and also the result that any Boolean ring is a commutative ring. The theorem to which we refer is, namely: any ring in which $x^{n(x)} = x$, $n(x)$ an integer greater than one, for every x in the ring, is necessarily commutative.

In this paper we prove a generalization of this result. In fact we prove the following: if in a ring $x^n - x$ is in the center for all ring elements x and a *fixed* integer n which is larger than 1, then the ring must be commutative. The proof is essentially composed of three parts: (1) the proof for semi-simple rings, (2) the location of the radical if the ring possesses one, (3) the proof for subdirectly irreducible rings, from which the general case follows.

The case $n = 2$.[1] Although the case for which $n = 2$ need not be singled out in order to make the proof go through in the general case, the proof in this instance is so simple and elementary that we feel it would be of some interest to present it by itself. Moreover, this situation might be considered as a direct generalization of the fact that a Boolean ring is commutative.

So let us assume that R is a ring with center C, and that for all $x \in R$, $x^2 - x \in C$. Suppose that $x, y \in R$. Thus $(x+y)^2 - (x+y) \in C$. Since both $x^2 - x$ and $y^2 - y$ are in C, we immediately obtain that $xy + yx \in C$; so $x(xy + yx) = (xy + yx)x$. Thus $x^2y = yx^2$, and so $x^2 \in C$. Since $x^2 - x$ is also in C we have that $x \in C$, which is the theorem in this particular case.

The theorem for semi-simple rings. Suppose that K is a division ring with center C and that there exists an integer n greater than 1 so that for all x in K, $x^n - x$ is in C. Suppose that $K \neq C$; that is, there is an $x \in K$, $x \notin C$. Hence (1) $x^n - x \in C$. Let c be any element in C; thus (2) $(cx)^n - cx = c^n x^n - cx \in C$. Combining (1) and (2) we obtain that

* Received January 17, 1951.

[1] The case $n = 2$ was conjectured, in a conversation, by Dr. Irving Kaplansky.

$(c^n - c)x \in C$; so if $c^n - c \neq 0$ it would have an inverse in C, and this would put x in C, contrary to supposition. So we must have that for all $c \in C$, $c^n = c$. Consequently C must be a *finite* field. Consider the sequence $\{c_i\}$ where $c_i = x^{n^i} - x$. All these c_i's are in C, and since C is finite, for some $i \neq j$, $c_i = c_j$. This leads to $x^s = x$ for some appropriate integer s, $s > 1$. So for all $x \in K$, $x^{n(x)} = x$, whence by Jacobson's theorem K is commutative. Thus we have shown

THEOREM 1. *If K is a division ring with center C, and if $x^n - x$ is in C for all x in R, and $n > 1$, then K is commutative.*

Before proceeding to the case of primitive and semi-simple rings we establish several easy, but essential, lemmas. We assume that R is a ring with center C and that n is an integer larger than 1.

LEMMA 1. *If for all $x \in R$, $x^n - x \in C$, then all the nilpotent elements of R are in C.*

Proof. Suppose that $x^t = 0$. Thus $x^{n^t} = 0$. Since $x^{n^t} - x \in C$ we have that x is in C.

LEMMA 2. *For the R of Lemma 1 all the idempotents are in the center.*

Proof. Suppose that $e^2 = e$. A simple calculation shows that for any $x \in R$, $(xe - exe)^2 = (ex - exe)^2 = 0$. Hence by Lemma 1 $xe - exe \in C$; thus $0 = e(xe - exe) = (xe - exe)e = xe - exe$. That is $xe = exe$; similarly $ex = exe$, whence $xe = ex$ and $e \in C$.

We consider next the situation where R is a primitive ring [3, p. 312] and where $x^n - x \in C$. Thus R possesses a maximal right-ideal ρ which contains no non-zero two-sided ideal of R. Let $x \in \rho$. Since $x^n - x \in C$, $(x^n - x)R$ is a two-sided ideal of R and is contained in ρ. Hence $(x^n - x)R = (0)$. Since R is primitive this forces $x^n = x$. Let $e = x^{n-1}$. A simple check verifies that $e^2 = e$, so e is in C by Lemma 2. Moreover, eR is then a two-sided ideal of R and is contained in ρ, whence $eR = (0)$, so $e = 0$. Now $x = ex = 0$, whence $\rho = (0)$. So (0) is a maximal right ideal in R, which is primitive, so R is a division ring, which by Theorem 1 is commutative; so we have

THEOREM 2. *If R is a primitive ring with center C, and if for all x in R, $x^n - x$ is in C, then R is commutative.*

Any ring, semi-simple in the sense of Jacobson, is isomorphic to a subdirect sum of primitive rings [3, Theorem 28]. So if R is semi-simple and

$x^n - x \in C$ for all $x \in R$, then R is isomorphic to a subdirect sum of primitive rings, each of which being homomorphic images of R inherit the property that $x^n - x$ is in the center. So, by Theorem 2, each of the components in this representation of R as a subdirect sum is commutative, hence R is itself commutative. Thus

THEOREM 3. *If R is a semi-simple ring and if for all x in R, $x^n - x$ is in the center, then R is commutative.*

The location of the commutators and radical. In what follows we assume that R is a ring with center C and enjoying the property that $x^n - x$ is in C for all x in R, with n a fixed integer larger than 1. Let N be the radical of R (that is, the Jacobson radical). Since R/N is semi-simple and in R/N $x^n - x$ is in the center, by Theorem 3, R/N is commutative. So we immediately obtain

THEOREM 4. *For all $x, y \in R$, $xy - yx \in N$.*

Suppose that $x \in R$, and $c \in C$. Thus (1) $x^n - x \in C$, (2) $(cx)^n - cx = c^n x^n - cx \in C$. Combining (1) and (2) we obtain

LEMMA 3. *For all $x \in R$, and $c \in C$, $(c^n - c)x \in C$.*

COROLLARY. *For all $x, y \in R$ and $c \in C$, $(c^n - c)(xy - yx) = 0$.*

Suppose that $a \in N \cap C$. Thus for all $x, y \in R$, $(a^n - a)(xy - yx) = 0$. Since $a \in N$, $1 - a^{n-1}$ (the unit element is purely formal) is not a divisor of zero. Thus $a \cdot (xy - yx) = 0$. Let $b \in N$. Thus $b^n - b \in N \cap C$, whence $(b^n - b)(xy - yx) = 0$. Repeating the argument as in the case of a, we have $b(xy - yx) = 0$ for all x and y in R. In particular $b(by - yb) = 0$. By the analogous argument, $(by - yb)b = 0$. From these two we obtain $b^2y = byb = yb^2$; that is, $b^2 \in C$. Now, $b^n - b$ is in C, so if n is even, since $b^2 \in C$, we would have $b^n \in C$ from which it would follow that b is in C. On the other hand, if n is odd, $1 - b^{n-1}$ would be in C, and is not a divisor of zero. This combined with $b(b^{n-1} - 1) \in C$ would put b in C. So in any case, $b \in C$. Thus

THEOREM 5. $N \subset C$.

Combining Theorems 4 and 5 we obtain

COROLLARY. *For all x, y in R, $xy - yx$ is in C.*

COROLLARY. *For all x, y in R, $(xy - yx)^2 = 0$.*

The theorem for general rings. We recall that a ring is said to be

subdirectly irreducible if the intersection of its non-zero two-sided ideals is not the zero ideal. [1, p. 764.] The importance of this concept is brought out in the theorem [1, Theorem 2].

THEOREM 6. *Every ring is isomorphic to a subdirect sum of subdirectly irreducible rings.*

Suppose that R is a ring in which $x^n - x \in C$ for all $x \in R$. R is isomorphic to a subdirect sum of subdirectly irreducible rings, each of which, being isomorphic images of R enjoy the property that $x^n - x$ is in the center. To prove our result it is thus sufficient to prove it for subdirectly irreducible rings. So we suppose that R is subdirectly irreducible and that for all $x \in R$, $x^n - x \in C$, the center of R. Let S be the intersection of the non-zero two-sided ideals of R. $S \neq (0)$.

THEOREM 7. *There exists a prime p so that for all $x, y \in R$, $p(xy - yx) = 0$.*

Proof. Since $x^n - x \in C$ and $(2x)^n - 2x \in C$, we easily obtain that for all $x \in R$, $(2^n - 2)x \in C$, and so for all $x, y \in R$, $(2^n - 2)(xy - yx) = 0$. So if R is not commutative it has non-zero elements of non-zero finite additive order. Let x, y be such that $xy - yx$ has minimal non-zero order p; clearly p is a prime. Let $R_p = \{x \in R \mid px = 0\}$. R_p is a two sided ideal of R, and is not (0), hence $R_p \supset S$. If $qx = 0$ for some $x \neq 0$, then $R_q \supset S$. So if q is relatively prime to p, since $mp + kq = 1$ for some integers m, k, we would obtain that $s = mps + kqs = 0$, for any $s \in S$, contrary to hypothesis. So any element in R having finite additive order has order a power of p. But since $(p^n - p)x \in C$, we obtain that $(p^{n-1} - 1)p(xy - yx) = 0$; thus $p(xy - yx) = 0$ for all $x, y \in R$.

COROLLARY. *For all $x \in R$, px annihilates all commutators.*

From the first corollary to Theorem 5 we have that $(xy - yx) \in C$. Thus $xy - yx = c$ where $c \in C$ and where $pc = 0$ by Theorem 7. Multiplying from the left by x we obtain $x^2y = xyx + xc = yx^2 + 2xc$. Continuing in this way we obtain that $x^iy = yx^i + icx^{i-1}$. Putting $i = p$, since $pc = 0$ we see that $x^py = yx^p$. Thus we have shown

THEOREM 8. *For every x in R, x^p is in C.*

Since $S \subset N$ we have that $S \subset C$. Since $S \subset$ commutator ideal, which is nilpotent (as can be seen by examining the proof of Theorem 5), $S^2 = (0)$. Let $A(S) = \{x \in R \mid xS = (0)\}$. Clearly $A(S)$ is a two-sided ideal of R,

and since it contains S, it is not the zero ideal. Two cases now present themself: (1) $A(S) = R$ and (2) $A(S) \neq R$.

Let us first consider the case that $A(S) = R$. By Theorem 8, for any $x \in R$, $x^p \in C$, so by the corollary to Lemma 3 we have $(x^{n^p} - x^p)(yz - zy) = 0$ for all $x, y \in R$. Thus

$$x^{n^p-p}x^p(yz - zy) = x^p(yz - zy).$$

Let $T = \{r \in R \mid x^{n^p-p}r = r\}$. Since $x^{n^p-p} \in C$, T is a two-sided ideal of R. T does not contain S, for $x^{n^p-p}S = (0)$. Thus $T = (0)$. Since $x^p(yz - zy) \in T$, we get that $x^p(yz - zy) = 0$ for all $y, z \in R$. In particular $x^p(xy - yx) = 0$. Thus $x^{p+1}y = x^pyx = yx^{p+1}$ since $x^p \in C$. Thus $x^{p+1} \in C$. Continuing in this way we obtain that $x^{p+k} \in C$ for any non-negative k. Since $x^n - x \in C$, $x^{n^p} - x \in C$. But since $n^p > p$, $x^{n^p} \in C$, so $x \in C$. Thus R is commutative; so

THEOREM 9. *If $A(S) = R$, then R is commutative.*

Let us now consider the situation when $A(S) \neq R$. Let $s \in S$, $s \neq 0$. Thus $sR = Rs$ (since $s \in S \subset C$) is a two-sided ideal of R and is contained in S. If $Rs = (0)$, then it would follow that $RS = (0)$ since S is a minimal two-sided ideal, and we have supposed that $RS \neq (0)$. Hence $Rs = S$. Suppose that $x \notin A(S)$. Thus $xs \neq 0$ for $s \neq 0$ in S; for if $xs = 0$, then $xsR = xS = 0$, a contradiction. Suppose $xs = t \neq 0$. Thus $Rt = S$, whence for some $y \in R$, $s = yt$. Thus $xyt = t$; in fact $xys = s$ for all $s \in S$ by the fact that $tR = S$. So $xy = e$ acts as a unit element on S. For any $x \notin A(S)$ we now can show that we can find a y so that $xy - e \in A(S)$ by the replica of the argument used in finding e. Thus $R/A(S)$ is a field. Hence

THEOREM 10. *$R/A(S)$ is a field.*

Our next goal is to establish that $R/A(S)$ is a finite field. To prove this we must first establish some side results. The first of these is

THEOREM 11. *If $a \in R$ is a divisor of zero then $a \in A(S)$.*

Proof. Suppose that $ax = 0$, $x \neq 0$. We maintain that we could have chosen $x \in C$. For if $x^p = 0$, then by Lemma 1, $x \in C$, and if $x^p \neq 0$, then by Theorem 8 $x^p \in C$ and $ax^p = 0$. So we suppose that $x \in C$. Thus Rx is a two-sided ideal of R. $Rx \neq (0)$ for if $Rx = (0)$, then $T = \{x \in R \mid Rx = 0\}$ would be a two-sided ideal of R, and not (0), so would contain S, and $RS = (0)$ contrary to hypothesis. So $Rx \neq (0)$, thus $Rx \supset S$, thus $aRx = axR = (0) \supset aS$, and $a \in A(S)$.

This enables us to prove that $R/A(S)$ is a finite field. For if $x \notin A(S)$,

since $x^p \in C$ by Theorem 8, we have that $(x^{np} - x^p)(yz - zy) = 0$ for all $y, z \in R$. Thus $x^{np} - x^p$ is a divisor of zero and must be in $A(S)$ by Theorem 11. In $R/A(S)$ we then have for every $\bar{x}$, $\bar{x}^{np} = \bar{x}^p$. Thus $R/A(S)$ is a finite field. By the corollary to Theorem 7, for all $x \in R$, $px \in A(S)$. So $R/A(S)$ is of characteristic p, and in it $\bar{x}^n = \bar{x}$. This gives

THEOREM 12. *$R/A(S)$ is a finite field of characteristic p.*

COROLLARY. $n \geq p$.

For in $R/A(S)$, $x^n = x$ for all x, and $n > 1$.

Using the fact that the multiplicative group of a finite field is cyclic we immediately obtain

THEOREM 13. *There exists an $a \in R$ so that for every $x \notin A(S)$ we can find an integer t so that $x - a^t \in A(S)$.*

If we could prove that $A(S)$ is in the center, this together with Theorem 13 would lead us quickly to the result we want; consequently the direction we take is to establish that $A(S) \subset C$. Our first result in this regard is

THEOREM 14. *If $x \in A(S)$, then x^p annihilates all commutators.*

Proof. Let $x \in A(S)$. Since $x^p \in C$, for all $y, z \in R$ we get, using Lemma 3, that $(x^{np} - x^p)(yz - zy) = 0$. Thus we have $x^{np-p}x^p(yz - zy) = x^p(yz - zy)$. Let $T = \{r \in R \mid x^{np-p}r = r\}$. Since $x^{np-p} \in C$, T is clearly a two-sided ideal of R. S is not in T since $x^{np-p}S = (0) \neq S$. So $T = (0)$, whence $x^p(yz - zy) = 0$, being an element of T. This proves the theorem.

Let $x \in A(S)$. Thus $x^p(xy - yx) = 0$. So $x^{p+1}y = x^pyx = yx^{p+1}$, since x^p is in the center. Thus $x^{p+1} \in C$. Continuing in this way we obtain that $x^{p+k} \in C$ for any non-negative integer k. Since $n \geq p$, we have that $x^n \in C$, which, combined with $x^n - x \in C$ yields that $x \in C$. So we have

THEOREM 15. $A(S) \subset C$.

Suppose that $x \notin A(S)$. By Theorem 13, $x - a^t \in A(S) \subset C$; so $x - a^t$ commutes with a hence x commutes with a. Since a also commutes with all the elements in $A(S)$, we see that a is in C. As a consequence of this, $a^t \in C$, which together with $x - a^t \in C$ puts x in C. So we have established

THEOREM 16. *If $A(S) \neq R$, then R is commutative.*

Theorems 9 and 16 cover between them the only two possibilities when R is subdirectly irreducible. So they yield

THEOREM 17. *If R is subdirectly irreducible and, for every x in R, $x^n - x$ is in the center of R, then R is commutative.*

Using Theorem 6 we obtain the generalization of Jacobson's theorem which we mentioned in the introduction, namely

THEOREM 18. *If R is a ring with center C, and if $x^n - x \in C$ for all x in R, n a fixed integer larger than 1, then R is commutative.*

Remarks. Before closing we should like to point out two other possible generalizations of Jacobson's theorem which should possibly be true: (1) If $x^{n(x)} - x \in C$ for all x in R, then R is commutative. (2) If R is a metric ring with a metric ρ and if for every x in R and any $\epsilon > 0$ we can find an integer $n(x, \epsilon) > 1$ so that $\rho(x^{n(x,\epsilon)}, x) < \epsilon$, then R is commutative.

Our original proof used the existence of a unit element. The referee kindly pointed out to us that to prove the theorem without the unit element substantially only involved the case division $A(S) \neq R$ and $A(S) = R$, the latter case not being possible, of course, in rings with unit elements.

THE OHIO STATE UNIVERSITY.

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A DEFINITION OF DEGREE FOR CERTAIN MAPPINGS IN HILBERT SPACE.*

By JANE CRONIN.

1. Introduction. The concept of the degree of a continuous mapping was first introduced by Brouwer [2, 1].[1] Brouwer defined degree for continuous mappings in Euclidean space. A definition of degree for certain mappings in non-locally compact spaces was introduced by Leray and Schauder [8]. They defined a degree for mappings in Banach space of the form $I + F$ where I is the identity transformation and F is completely continuous. Their work left open the question of whether a degree could be defined for all continuous mappings in a Banach space. Leray [7] and Kakutani [6] showed that the answer to this question is no, even if the Banach space is a Hilbert space.

The purpose of this paper is to define a degree for mappings in Hilbert space of the form $I + C + T$, where I is the identity, C is linear, self-adjoint, and completely continuous, and T is a higher order transformation which need not be completely continuous.

In Section 2, the definition of the degree is given and it is proved that this degree has the usual properties of a mapping degree. It is shown that this degree has the same value as the Leray-Schauder degree of $I + C + T$ if the Leray-Schauder degree (hereafter to be termed the LS degree; the Leray-Schauder index [8, p. 54] will be termed the LS index) is defined for $I + C + T$.[2] Also a method for computing the value of the degree for a given mapping is described. The degree defined is essentially the same as the "multiplicity" defined in a previous paper [3]. The real content of Section 2 is the proof that the multiplicity defined in [3] has the properties of a mapping degree, especially that the degree is invariant under homotopy.

The degree is defined only relative to "sufficiently small" sets[3] and in proving the invariance under homotopy of the degree, we assume that the di-

* Received May 17, 1950.

[1] Numbers in brackets refer to the bibliography at the end.

[2] In general, the LS degree is not defined for the mapping $I + C + T$ because T is not, in general, completely continuous.

[3] When we refer to the degree at a point p of a mapping M relative to a set S, we mean the degree at p of M/S, i.e., the mapping M considered on the set S.

mension of the null space of $I+C$ is constant during the homotopy. In Section 3, we exhibit an example which justifies the introduction of the degree. That is, we describe a transformation of $L_2(0,1)$ into itself which is of the form $I+C+T$ and hence can be treated by our theory but which is such that its higher order part is not completely continous; so that it is impossible to treat the mapping using the Leray-Schauder theory.

The multiplicity introduced in [3] is actually defined for mappings in Banach space. We are obliged to restrict ourselves to mappings in Hilbert space here in order to prove the invariance under homotopy. Precisely where the Hilbert space theory is used is indicated in footnote [6].

2. Definition of the mapping degree. Let $\mathfrak{X}$ be a Hilbert space. Consider the mapping of $\mathfrak{X}$ into itself $I+C+T$, where I is the identity transformation on $\mathfrak{X}$, C is a linear, self-adjoint completely continuous mapping of $\mathfrak{X}$ into itself and T is a mapping of a subset of $\mathfrak{X}$ into $\mathfrak{X}$ with the following properties:

(P_1) $T(0)=0$;

(P_2) there exist a neighborhood N of 0 and a positive constant B such that if $x_1, x_2 \in N$, then $\| T(x_1)-T(x_2)\| \leq B[\| x_1 \| + \| x_2 \|][\| x_1 - x_2 \|]$.

Suppose first that $I+C$ is nonsingular, i. e., takes only 0 into 0. Then from the Riesz theory of operators of the form $I+C$ [10], it follows that $(I+C)^{-1}$ exists. Hence the LS index of $I+C$ at 0 is defined and is 1 or -1 (see [8, pp. 56-58]). Now we consider the equation $(I+C+T)(x)=(y)$. Applying $(I+C)^{-1}$ to this equation, we obtain

$$x+(I+C)^{-1}T(x)=(I+C)^{-1}(y). \tag{2.1}$$

From properties P_1 and P_2 of T, it follows that there exist ϵ_1 and ϵ_2 such that if $\| x \| < \epsilon_1$ and $\| y \| < \epsilon_2$, then the Hildebrandt-Graves implicit function theorem [5, Theorem 2] can be applied to solve (2.1) uniquely for x in terms of y. This suggests that we define the mapping degree of $I+C+T$ at y such that $\| y \| < \epsilon_2$ and relative to the sphere $S=[x/\| x \| < \epsilon_1]$[3] to be 1. Actually, to avoid the ambiguity of sign that results from the multiplication by $(I+C)^{-1}$, we define the mapping degree of $I+C+T$ at y such that $\| y \| < \epsilon_2$ and relative to S to be the LS index of $I+C$ at 0. (Pairs $\epsilon'_1 > 0$, $\epsilon'_2 > 0$ such that $\epsilon'_2 \leq \epsilon_1$ and $\epsilon'_2 \leq \epsilon_2$ may be used in place of ϵ_1 and ϵ_2).

Suppose now that $I+C$ is singular, i. e., that there exists $z \in \mathfrak{X}$, $z \neq 0$ such that $(I+C)(z)=0$. In [10], it is shown that $\mathfrak{X}_1$, the null space of $I+C$, is a finite dimensional linear space. Also, we may write $\mathfrak{X}=\mathfrak{X}_1 \oplus \mathfrak{X}^1$

(direct sum), where $\mathfrak{X}_1$ and $\mathfrak{X}^1$ are orthogonal and on $\mathfrak{X}^1$, $I+C$ is a 1-1 transformation taking $\mathfrak{X}^1$ onto itself. These properties follow from [10] and the fact that C is self-adjoint. If E_1, E^1 are the projections of $\mathfrak{X}$ onto $\mathfrak{X}_1$ and $\mathfrak{X}^1$ respectively, then it is easy to show that the transformation $R=(I+C+E_1)^{-1}$ has the property: $R(I+C)x=E^1(x)$. Moreover, R is of the form $I+G$ where I is the identity and G is linear and completely continuous. Since R is nonsingular and of this form, the LS index at 0 of R is defined.

Now we consider the equation

$$(*) \qquad (I+C+T)(x)=y;$$

multiplying by R, we obtain:

$$(2.2) \qquad E^1(x)+RT(x)=R(y).$$

Now we apply E_1 and E^1 respectively to (2.2) and obtain

$$(2.3) \qquad E_1RT(x_1+x^1)=E_1R(y),$$

where x_1 and x^1 denote $E_1(x)$ and $E^1(x)$ respectively, and

$$(2.4) \qquad x^1+E^1RT(x_1+x^1)=E^1R(y).$$

It is clear that solving (2.2) (and therefore (*)) for x when y is given is equivalent to solving (2.3) and (2.4) simultaneously for x_1 and x^1 when y is given.

From properties P_1 and P_2 of T, it follows that there exist $\epsilon_3>0$, $\epsilon_4>0$, $\epsilon_5>0$ such that if $\|x^1\|<\epsilon_3$, $\|x_1\|<\epsilon_4$, $\|y\|<\epsilon_5$, then the Hildebrandt-Graves implicit function theorem can be applied to solve (2.4) uniquely for x^1 in terms of x_1 and y. That is, we have:

$$(2.5) \qquad x^1=H(x_1,y),$$

where H is uniformly continuous in the subspace of the product space $\mathfrak{X}_1\times\mathfrak{X}$ in which it is defined. Substituting from (2.5) into (2.3), we obtain:

$$(2.6) \qquad E_1RT[x_1+H(x_1,y)]-E_1R(y)=0.$$

Thus the problem of solving (2.2) for x when y is given is reduced to that of solving (2.6), an equation in the finite-dimensional space $\mathfrak{X}_1$, for x_1 when y is given. This suggests defining the degree of the mapping $I+C+T$ at a point $y=y_0$ such that $\|y_0\|<\epsilon_5$ and relative to the sphere $[x/\|x\|<\epsilon_6]$ where ϵ_6 is such that if $\|x\|\leq\epsilon_6$, then $\|x^1\|<\epsilon_3$ and $\|x_1\|<\epsilon_4$, to be the topological degree at 0 of the mapping in $\mathfrak{X}_1$:

$$(2.7) \qquad E_1RT[x_1+H(x_1,y_0)]-E_1R(y_0).$$

Actually, in order to avoid the ambiguity of sign resulting from multiplying the original equation by R, we define the degree to be the product of the degree at 0 of (2.7) and the LS index at 0 of R.

Since the Hildebrandt-Graves theorem yields only a local solution, we must restrict our considerations to small values of x and y. In [3], it was shown that there exist ϵ_7 and ϵ_8 such that if $\| x \| < \epsilon_7$ and $\| y \| < \epsilon_8$, then the Hildebrandt-Graves theorem can be applied and the topological degree of (2.7) computed.[4] It was also shown that the degree is the same for all y such that $\| y \| < \epsilon_8$. We summarize this discussion in a formal definition of the mapping degree. Let $\epsilon \leq \epsilon_7$, $\epsilon' \leq \epsilon_8$. Then we have:

Definition 2.1. If $I + C$ is nonsingular, the mapping degree of $I + C + T$ at y such that $\| y \| < \epsilon'$ and relative to the sphere $S = [x/\| x \| \leq \epsilon]$ is the LS index at 0 of $I + C$. If $I + C$ is singular, the mapping degree of $I + C + T$ at y such that $\| y \| < \epsilon'$ and relative to S is the product of the LS index at 0 of R and the topological degree of mapping (2.7) at 0, relative to the sphere $[x_1/\| x_1 \| \leq \epsilon_0]$ in $\mathfrak{X}_1$, where ϵ_0 depends on ϵ as described in [3].[5] We will denote this mapping degree by $d[I + C + T, S, y]$.

Now we show that the mapping degree just defined has some of the usual properties of a mapping degree. First since the degree is defined as the product of the LS index of R and the degree of the mapping

$$E_1 RT[x_1 + H(x_1, y)] - E_1 R(y)$$

at 0 and relative to the sphere $[x_1/\| x_1 \| \leq \epsilon_0]$, then the mapping degree for sufficiently small y exists if for x_1 such that $\| x_1 \| = \epsilon_0$,

$$E_1 RT[x_1 + H(x_1, 0)] \neq 0.$$

Hence the condition that the mapping degree exists is different from the usual condition that the degree of the mapping at y relative to S is defined if no point of the boundary of S is taken into y by the mapping. Instead we have:

THEOREM 2.1. *The mapping degree $d[I + C + T, S, y]$ exists if and only if for each $x_0 \in S$ such that $\| E_1(x_0) \| = \epsilon_0$, we have $(I + C + T)(x_0) \neq 0$.*

---

[4] In computing the topological degree, properties P_3 and P_4 (stated later on just before Lemma 2.1) are used. We choose ϵ_7 and ϵ_8 small enough so that P_3 and P_4 can be utilized as described in [3].

[5] From a topological viewpoint, it would be better to call this mapping degree the index of $I + C + T$ at 0. However, in the application, we are interested in nearby points as well as 0 itself. Hence it is more convenient to think of a mapping degree than an index.

Proof. Suppose first that $I + C$ is singular. Then if

$$(I + C + T)x_0 = 0, \tag{2.8}$$

we have, applying R:

$$E^1(x_0) + RT[E_1(x_0) + E^1(x_0)] = 0.$$

Applying E^1 and E_1, we obtain

$$E^1(x_0) + E^1RT[E_1(x_0) + E^1(x_0)] = 0 \tag{2.9}$$

and

$$E_1RT[E_1(x_0) + E^1(x_0)] = 0. \tag{2.10}$$

Solving (2.9), we have $E^1(x_0) = H[E_1(x_0), 0]$. Substituting in (2.10), we have

$$E_1RT\{E_1(x_0) + H[E_1(x_0), 0]\} = 0. \tag{2.11}$$

Hence the mapping degree is not defined. Reversing the steps in this proof shows that equation (2.11) implies equation (2.8). If $I + C$ is nonsingular, the condition is trivially fulfilled because E_1 is the zero transformation in that case. From the definition, it is known that if $I + C$ is nonsingular, the degree is always defined. This completes the proof of Theorem 2.1.

THEOREM 2.2. *The mapping degree $d[I + C + T, S, y]$ has the following properties:*

(i) *If $d[I + C + T, S, y] \neq 0$, there exists $x \in S$ such that $(I + C + T)(x) = y$.*

(ii) *$d[I + C + T, X, y]$ is constant for all y sufficiently close to zero.*

Proof. Property (i) follows directly from the definition of $d[I + C + T, S, y]$ and the corresponding property of the topological degree of the mapping in Euclidean space. If $I + C$ is nonsingular, property (ii) follows from the definition of $d[I + C + T, S, y]$. As remarked before, the proof for the case when $I + C$ is singular is contained in [3, Section 3]. This completes the proof of Theorem 2.2.

THEOREM 2.3. *If T is completely continuous and if $d[I + C + T, S, 0]$ exists, then the* LS *degree of $I + C + T$ at 0 and relative to S exists and is equal to $d[I + C + T, S, 0]$.*

Proof. See [3, Section 5].

Now we prove the invariance under homotopy of $d[I + C + T, S, y]$. We state first what we mean by a homotopy in this work.

Definition 2.2. A homotopy $I(x,t) + C(x,t) + T(x,t)$ is a continuous mapping from $\mathfrak{X} \times \mathfrak{J}$, where $\mathfrak{J}$ is the closed unit interval $(0,1)$ into $\mathfrak{X}$ satisfying these conditions:

(1) $I(x,t) = x$ for each pair $(x,t) \varepsilon \mathfrak{X} \times \mathfrak{J}$.

(2) For each fixed t, $C(x,t)$ is a linear, self-adjoint, completely continuous transformation of $\mathfrak{X}$ into itself. Also $C(x,t)$ is a continuous function of t in the following sense: for arbitrary $t_0 \varepsilon \mathfrak{J}$, we have $\lim_{t_1 \to t_0} \| C_{t_1} - C_{t_0} \| = 0$, where C_{t_1} denotes the transformation of $\mathfrak{X}$ into $\mathfrak{X}$ obtained by taking $t = t_1$ in $C(x,t)$ and C_{t_0} is defined similarly.

(3) $T(x,t)$ is a continuous mapping from $\mathfrak{X} \times \mathfrak{J}$ into $\mathfrak{X}$ with the property: for each fixed $t_0 \varepsilon \mathfrak{J}$, $T(x,t)$ is a mapping of a subset of $\mathfrak{X}$ into $\mathfrak{X}$ with properties P_1 and P_2. Moreover the neighborhood N and the constant B are independent of t.

THEOREM 2.4. *If $I(x,t) + C(x,t) + T(x,t)$ is a homotopy satisfying the conditions of Definition 2.2 and if the following additional conditions are satisfied:*

(4) *The dimension of the null space of $I(x,t_0) + C(x,t)$ is the same for all $t_0 \varepsilon \mathfrak{J}$.*

(5) *For each $t_0 \varepsilon \mathfrak{J}$ and for a sphere S which is independent of T, $d[I(x,t_0) + C(x,t_0) + T(x,t_0), S, y]$ is defined for y such that $\| y \| < \eta$, where η is independent of t.*

Then the mapping degree $d[I(x,t_0) + C(x,t_0) + T(x,t_0)]$ is the same for all $t_0 \varepsilon \mathfrak{J}$.

Proof. If $I + C$ is nonsingular, the proof follows from the fact that the LS degree of $I + C$ is invariant under homotopy [11, Theorem 2]. So we need only consider the case when $I + C$ is singular. From Definition 2.1 it follows that the mapping degree of $I + C + T$ is the same as the mapping degree of

$$x - E_1(x) + E_1 RT[x_1 + H(x_1, 0)] \tag{2.12}$$

except for a factor ± 1. For the mapping degree of (2.12) is equal to the topological degree at 0 of

$$E_1 RT[x_1 + H(x_1, 0)] \tag{2.13}$$

and $d[I + C + T, S, y]$ for y sufficiently close to 0 is equal to the topological

degree of (2.13) multiplied by the LS index at 0 of R, which is 1 or -1. Now (2.12) may be written:

$$x - E_1(x) + E_1RT\{E_1(x) + H[E_1(x), 0]\}. \tag{2.14}$$

Since E_1 is finite-dimensional and hence completely continuous, it is clear that the LS degree of (2.14) is defined, i.e., (2.14) is a mapping of the form $I + F$ where F is completely continuous.

Now let E_1^t be the projection of X onto the null space of $I + C_t$ (for convenience we write the homotopy parameter as a subscript, i.e., we denote $C(x, t)$ by $C_t(x)$ and $T(x, t)$ by $T_t(x)$ and let R_t be the linear transformation such that $R_t(I + C_t)x = x - E_1^t(x)$. We let $H_t[E_1^t(x), y]$ denote the function obtained by applying Hildebrandt-Graves Theorem.) We will show that

$$x - E_1^t(x) + E_1^tR_tT_t\{E_1^t(x) + H_t[E_1^t(x), 0]\} \tag{2.15}$$

and R_t are homotopies in the sense of [11, Definition 4]. Then the LS degrees are the same for all t such that $0 \leq t \leq 1$ [11]. Since for each t, the mapping (2.15) is a layer mapping [11], it LS degree is by definition the topological degree of $E_1^tR_t\{E_1^t(x) + H_t[E,^t(x), 0]\}$. The proof of Theorem 2.4 then follows from the relation of the mapping degrees of $I + C + T$ and (2.12) described above.

Hence to complete the proof of Theorem 2.4, it suffices to prove that E_1^t, R_t, and H_t are continuous mappings from $\mathfrak{X} \times \mathfrak{J}$ into $\mathfrak{X}$ and that the mapping

$$E_1^t\{-x + R_tT_t\{E_1^t(x) + H_t[E_1^t(x), 0]\}\}$$

takes any bounded set in $\mathfrak{X} \times \mathfrak{J}$ into a sequentially compact set in $\mathfrak{X}$. (Cf. [11, Definition 4].) This last statement is an immediate consequence of the fact that any bounded set $V = (y_\nu, t_\nu)$ of $\mathfrak{X} \times \mathfrak{J}$ each of whose points satisfy the equation $-y_\nu = C_{t_\nu}(y_\nu)$ is sequentially compact in $\mathfrak{X} \times \mathfrak{J}$. To prove this: let t_0 be a limit point of the set $\{t_\nu\}$. Then there is a subsequence y_{ν_n} such that $\lim_{n\to\infty} C_{t_0}(y_{\nu_n})$ exists and is some point, say y_0. This is because C_{t_0} is completely continuous and $\{y_\nu\}$ is bounded. Then by (2) of Definition 2.2, there exists $\lim_{n\to\infty} C_{t_{\nu_n}}(y_{\nu_n}) = -\lim_{n\to\infty} y_{\nu_n} = y_0$.

Now we need to show that E_1^t is a continuous mapping from $\mathfrak{X} \times \mathfrak{J}$ into $\mathfrak{X}$.[6] More precisely, we want to show $\lim_{t\to t_0} \| E_1^t - E_1^{t_0} \| = 0$. This is a

[6] It is for the proof of this statement that Hilbert space theory is used here.

direct consequence of [9, Satz 5]. By definition, $R_t = (I + C_t + E_1^t)^{-1}$; so R_t is a continuous mapping from $\mathfrak{X} \times \mathfrak{J}$ into $\mathfrak{X}$.

Finally, we must show that H_t depends continuously on t. $H_t[E_1^t(x), y]$ is the solution of the equation

$$E_t^1(x) + E_t^1 R_t T_t[E_1^t(x) + E_t^1(x)] = E_t^1 R_t(y) \tag{2.17}$$

for $E_t^1(x)$ in terms of y and $E_1^t(x)$ which is obtained by using the implicit function theorem of Hildebrandt and Graves [5, Theorem 2]. For fixed t_0, the initial solution of (2.17) is $E_t^1(x) = E_1^t(x) = y = 0$, and $H_t[E_1^t(x), y]$ is the limit of the sequence:

$$H_{t_0}^{(1)}[E_1^{t_0}(x), y] = E_{t_0}^1 R_{t_0} T_{t_0}[E_1^{t_0}(x)] - E_{t_0}^1 R_{t_0}(y)$$

$$H_{t_0}^{(2)}[E_1^{t_0}(x), y] = E_{t_0}^1 R_{t_0} T_{t_0}\{E_1^{t_0}(x) + H_{t_0}^{(1)}[E_1^{t_0}(x), y]\} - E_{t_0}^1 R_{t_0}(y)$$

$$\cdots$$

$$H_{t_0}^{(n)}[E_1^{t_0}(x), y] = E_{t_0}^1 R_{t_0} T_{t_0}\{E_1^{t_0}(x) + H_{t_0}^{(n-1)}[E_1^{t_0}(x), y]\} - E_{t_0}^1 R_{t_0}(y).$$

In [5], it is proved that the sequence $H_{t_0}^{(n)}[E_1^{t_0}(x), y]$ is a Cauchy sequence for each fixed pair $[E_1^{t_0}(x), y]$ and that the convergence is uniform in $[E_1^{t_0}(x), y]$. To prove the desired statement, we show that the convergence is uniform in t. Since $\| E_1^t \|$, $\| E_t^1 \|$, and $\| R_t \|$ are bounded in t, and since by (3) of Definition 2.2, the bound B of T_t is independent of t, the same type of argument as is used in [5] can be used to show that the convergence is uniform in t. This completes the proof of Theorem 2.4.

Now we describe a method for computing the degree of a given mapping. In order to do this, we impose two more conditions on the transformation. We assume the transformation T has the following properties:

(P_3) The transformation T is split into a term of order $k (\geq 2)$ and a term of higher order; that is, $T(x) = T^{(k)}(x) + \tilde{T}^{(k+1)}(x)$ where $T^{(k)}$ and $\tilde{T}^{(k+1)}$ are both continuous and:

(1) $T^{(k)}$ is homogeneous of degree k; that is, if m is an arbitrary integer, then

$$T^{(k)}\left(\sum_{i=1}^{m} a_i x_i\right) = \sum_{(b_i)} a_1^{b_1} \cdots a_m^{b_m} T_{b_1 \cdots b_m}[x_1, \cdots, x_m],$$

where the summation is over all sets of non-negative integers $[b_1, \cdots, b_m]$ such that $\sum_{i=1}^{m} b_i = k$ and $T_{b_1 \cdots b_m}$ is a continuous mapping from $\mathfrak{X} \times \cdots \times \mathfrak{X}$ (m times) into $\mathfrak{X}$.

(2) The $\lim_{x \to 0} \| \tilde{T}^{(k+1)}(x) \| / \| x \|^k = 0$.

(P_4) $E_1 R T^{(k)}(x_1) \neq 0$ for all $x_1 \in \mathfrak{X}_1$ such that $x_1 \neq 0$.

LEMMA 2.1. *If $I + C$ is singular and if T has properties P_1, P_2, P_3, and P_4, then the mapping degree of $I + C + T$ is equal to the topological degree at zero of a mapping of Euclidean n-space E^n (real or complex as $\mathfrak{X}$ is real or complex) given by n polynomials in n variables, the polynomials homogeneous of degree k in the n variables, where n is the dimension of the null space of $I + C$ and k is the "order" of T described by P_3.*

Proof. See [3, Section 3].

The degree of such a mapping described by homogeneous polynomials can be computed. We have:

THEOREM 2.5. *The topological degree at zero of the type of polynomial mapping described in Lemma 2.1 is k^n if E^n is complex and is $\equiv k^{n-1} \pmod 2$ if E^n is real.*

The proof of Theorem 2.5 will be given in another paper.

3. An example. In order to justify the definition of mapping degree given in Section 2, we must show that it is independent of previous work. That is, we must show that our theory is not contained in the Leray-Schauder theory. So we must exhibit a mapping $I + C + T$ which satisfies our hypotheses but which is not of the form $I + F$ where F is completely continuous. Since C is completely continuous, we must give an example of a mapping T which has properties P_1 and P_2 and which is not completely continuous.

A simple such example is the following: Let $\mathfrak{X}$ be the Hilbert space of square Lebesgue integrable functions $x(t)$ over the group of real numbers mod 1, i.e., $\mathfrak{X} = L_2(0, 1)$. We define

$$T[x(t)] = \int_0^1 x(t) x(s - t)\, dt$$

i.e., $T[x(t)]$ is the convolution of x with itself. It is a straightforward verification to show that this T has properties P_1 and P_2. The transformation T is not completely continuous because the bounded set $\{e^{2\pi i n t}\}$ is taken into itself and it is clear that $\{e^{2\pi i n t}\}$ contains no convergent subsequence.

Examples of the transformation C abound. Any linear integral transformation acting on $x(t) \in L_2(0, 1)$,

$$\int_0^1 K(x, t) x(t)\, dt,$$

where $K(s,t) = \overline{K(t,s)}$ and $K(s,t)$ is continuous simultaneously in s and t, is self-adjoint and completely continuous. Hence as a simple example of a transformation to which our theory may be applied but to which the Leray-Schauder theory cannot be applied, we have the transformation

$$x(s) + \int_0^1 K(s,t)x(t)\,dt + \int_0^1 x(t)x(s-t)\,dt$$

acting on $x(s)\,\varepsilon\, L_2(0,1)$.

ANN ARBOR, MICHIGAN.

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ON APPROXIMATION BY NON-DENSE SETS OF TRANSLATES.*

By I. I. HIRSCHMAN, JR.

We denote by $L_2(-\infty,\infty)$ the Hilbert space of complex valued functions $f(x)$ defined for $-\infty < x < \infty$ and such that

$$\| f(x) \| = [\int_{-\infty}^{\infty} | f(x) |^2 dx]^{1/2}$$

is finite. It has been shown by Laurent Schwartz [4], that if $t_1 < t_2 < t_3 < \cdots$ is a sequence of real numbers such that

$$\sum_{n=1}^{\infty} e^{-t_n} < \infty, \liminf_{n\to\infty} (t_{n+1} - t_n) e^{t_n} > 0,$$

and if $f(x)$ lies in the closure in $L_2(-\infty,\infty)$ of the subspace of finite linear combinations of the functions

$$e^{\frac{1}{2}(t_n - x)} \exp[-e^{(t_n - x)}], \qquad (n = 1, 2, \cdots),$$

i. e., if given $\epsilon > 0$ there is an integer N and constants $a_1, a_2, \cdots, a_N$ such that

$$\| f(x) - \sum_{n=1}^{N} a_n e^{\frac{1}{2}(t_n - x)} \exp[-e^{(t_n - x)}] \| \leqq \epsilon,$$

then $f(x)$ can be represented as a series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n e^{\frac{1}{2}(t_n - x)} \exp[-e^{(t_n - x)}]$$

for almost all x. In the present paper we shall establish a general theorem of this type.

Let $\Phi(x) \in L_2(-\infty,\infty)$ and $\phi(t) \in L_2(-\infty,\infty)$ be Fourier transforms of each other.

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{itx} \phi(t)\, dt \qquad (M_2),$$

(1)

$$\phi(t) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-itx} \Phi(x)\, dx \qquad (M_2).$$

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The symbol M_2 indicates that the integral in question is to be taken as the limit in the mean of order two of the partial integrals $\int_{-T}^{T}$ as T increases to infinity. Let $\nu(t)$ be defined for $0 \leqq t < \infty$ and such that

$$\nu(0) > 0, \qquad \nu'(t) \geqq 0, \qquad (0 \leqq t < \infty), \tag{2}$$

$$t\nu'(t)/\nu(t) = o(1) \text{ as } t \to +\infty, \int_1^\infty [t\nu(t)]^{-1} dt = \infty.$$

We define $H(t)$ and $\mathfrak{H}(t)$ by the equations

$$H(t) = (2/\pi) \int_1^t [u\nu(u)]^{-1} du, \tag{3}$$

$$t = (2/\pi) \int_1^{\mathfrak{H}(t)} [u\nu(u)]^{-1} du.$$

It is supposed that $\{t_n\}_1^\infty$ is an increasing sequence of real numbers.

THEOREM. *Let $\Phi(x)$, $\phi(t)$, $\nu(t)$, $H(t)$, $\mathfrak{H}(t)$, and $\{t_n\}_1^\infty$ be defined as above. If*

1. $|\phi(t)| \geqq \exp[-|t|/\nu(|t|)]$ *for all t,*
2. $|\Phi(x)| \leqq \exp[-\mathfrak{H}(x)]$ *for large positive x.*
3. $\liminf_{n\to\infty} (t_{n+1} - t_n) n\nu(n) > 0$,
4. $\liminf_{n\to\infty} t_n/H(n) > 1$,

and if $f(x)$ lies in the closure of the subspace of $L_2(-\infty, \infty)$ formed by the finite linear combinations of the functions $\{\Phi(t_n - x)\}_1^\infty$, then $f(x)$ may be represented as a series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \Phi_n(t_n - x)$$

for almost all x.

Our proof depends upon the construction of a sequence of functions $\{\Psi_n(x)\}_1^\infty$ such that

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} \Psi_n(x) \Phi(t_m - x)\, dx = \delta_{n,m}, \tag{4}$$

$$\| \Psi_n \| = O(\exp \mathfrak{H}(\mu t_n)), \qquad n \to \infty, \mu < 1. \tag{5}$$

Suppose that we have exhibited a sequence of continuous functions $\Lambda_n(t)$, $n = 1, 2, \cdots$, defined for $-\infty < t < \infty$ and belonging to $L_2(-\infty, \infty)$, such that if $\lambda_n(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{itu}\Lambda_n(t)\,dt$ then

$$\Lambda_n(t_m) = \delta_{n,m}, \tag{6}$$

$$\| \lambda_n(u)/\phi(u) \| = O(\exp \mathfrak{H}(\mu t_n)), \qquad n \to \infty, \mu < 1. \tag{7}$$

We set $\Psi_n(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-itu}[\lambda_n(u)/\phi(u)]\,du$ (M_2). It is asserted that the functions $\{\Psi_n\}_1^{\infty}$ so defined satisfy conditions (4) and (5). By Parseval's equality we have

$$\int_{-\infty}^{\infty} \Psi_n(x)\Phi(t_m - x)\,dx = \int_{-\infty}^{\infty} [\lambda_n(u)/\phi(u)][\phi(u)e^{-it_m u}]\,du$$

$$= \int_{-\infty}^{\infty} \lambda_n(u)e^{-iut_m}du = (2\pi)^{\frac{1}{2}}\Lambda_n(t_m) = (2\pi)^{\frac{1}{2}}\delta_{n,m}.$$

Using Parseval's equality again we obtain

$$\| \Psi_n(x) \| = \| \lambda_n(u)/\phi(u) \| = O(\exp \mathfrak{H}(t_n\mu)), \qquad n \to \infty.$$

We proceed to the construction of the functions $\{\Lambda_n(t)\}_1^{\infty}$. By assumption 4 there exists a number r, $0 < r < 1$, such that for n sufficiently large $t_n \geqq H(n)/r^2$. We define $\psi(t)$, $\mathfrak{H}^*(t)$, $R(t)$ and $\rho(t)$ by the equations:

$$\begin{aligned} &\psi(t) = \nu(t)r^2, \qquad t = (2/\pi)\int_1^{\mathfrak{H}^*(t)} [u\psi(u)]^{-1}du, \\ &R(t) = \log \mathfrak{H}^*(t), \qquad \rho(t) = \psi[\mathfrak{H}^*(u)]. \end{aligned} \tag{8}$$

Let S_z be the strip in the z plane, $z = x + iy$, defined by the inequality $|y| < \pi/2$ for $-\infty < x < \infty$, and let S_w be the curvilinear strip in the w plane, $w = u + iv$, defined by the inequalities $|v| < r/\rho(0)$ for $-\infty < u < 0$, $|v| < r/\rho(u)$ for $0 \leqq u < \infty$. Let $\chi(w)$ be the function which maps S_w conformally upon S_z and is normalized by the conditions $\chi(0) = 0$, $\chi'(0) > 0$.

We set

$$\begin{aligned} &d = \exp\,(4\pi r)/\cos\,(\pi r/2) \\ &m(u) = \exp\,\{-d\exp[r\chi(u)]\}/[4\rho(0)^2 + u^2]. \end{aligned} \tag{9}$$

Let $A^*(z)$ be analytic and $|A^*(z)| \leqq 1$ in S_z. We define

$$A(u) = m(u)A^*[\chi(u)]$$

$$\alpha(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} A(u)e^{itu}du, \qquad -\infty < t < \infty. \tag{10}$$

It has been shown in [2, § 5] that under these assumptions

$$|\alpha(t)| \leqq c \exp[-|t|/r\nu(|t|)], \qquad -\infty < t < \infty, \tag{11}$$

where c is a constant depending on $\nu(t)$ and r, but independent of $A^*(z)$.

Using (2) and (8) it is easily verified that

$$(d/dt)[r/\rho(t)] = -(r\pi/2)\mathfrak{H}^*(t)\nu'[\mathfrak{H}^*(t)]/\nu[\mathfrak{H}^*(t)] = o(1),$$

as $t \to +\infty$. It follows from [3; § 2] that

$$\chi(t) \sim R(t)/r, \qquad t \to +\infty,$$

$$\lim_{t\to+\infty} \chi(t + \rho(t)^{-1}wr) - \chi(t) = (\pi/2)w, \tag{12}$$

uniformly for $|w| \leqq 1 - \epsilon$, $\epsilon > 0$.

Let $x_n = \chi(t_n)$, $n = 1, 2, \cdots$. By (12) we have $x_n \sim R(t_n)/r$ as $n \to \infty$. Now $R(t) = \log \mathfrak{H}^*(t)$, $\mathfrak{H}^*(t) = \mathfrak{H}(r^2t)$, $t_nr^2 \geqq H(n)$ for large n, and $\mathfrak{H}(t)$ is increasing. It follows that x_n asymptotically exceeds $\log n/r$. From this we deduce that $\Sigma e^{-x_n} < \infty$. By assumption 4 lim inf $n\nu(n)[t_{n+1} - t_n] > 0$ as $n \to \infty$. We have $\rho(t_n) = r^2\nu[\mathfrak{H}(r^2t_n)]$. Since $t_n \geqq r^{-2}H(n)$ for large n and since $\mathfrak{H}(t)$ is increasing we have $\rho(t_n) \geqq r^2\nu(n)$. Thus as $n \to \infty$ lim inf $n\rho(t_n)[t_{n+1} - t_n] > 0$. It follows from (12) that lim inf $n[x_{n+1} - x_n] > 0$. Using the fact that x_n asymptotically exceeds $\log n/r$, we obtain finally that lim inf $(x_{n+1} - x_n)e^{x_n} > 0$.

We define

$$L^*_n(z) = \prod_{m=1}^{\infty}{}' [e^z - x_m]/[e^z + e^{x_m}], \ (m \neq n), \quad n = 1, 2, \cdots,$$

Since $\sum_{n=1}^{\infty} e^{-x_n} < \infty$, we have $L^*(z)$ analytic and $|L^*(z)| \leqq 1$ in S_z. See [4, § 9, and § 10]. We put

$$L_n(u) = m(u)L^*_n[\chi(u)], \qquad -\infty < u < \infty. \tag{13}$$

We may now define the functions $\{\Lambda_n\}_1^{\infty}$. We set

$$\Lambda_n(t) = L_n(t)/L_n(t_n), \qquad n = 1, 2, \cdots. \tag{14}$$

We must verify properties (6) and (7). Property (6) is evident. To

establish (7) we note that the relation $\liminf_{n\to\infty}[x_{n+1}-x_n]e^{x_n}>0$ implies that $|L^*{}_n(x_n)|\geqq\exp[-\exp(x_n-a)]$ for any a, provided n is sufficiently large. See [4, § 9 and § 10]. Taking $a=0$ we have $L^*{}_n[\chi(t_n)]\geqq\exp[-\exp x_n]$ for n sufficiently large. Now $x_n\sim R(t_n)/r$ as $n\to\infty$, $R(t_n)=\log\mathfrak{H}^*(t_n)$, $\mathfrak{H}^*(t_n)=\mathfrak{H}(r^1t_n)$. Thus $\log\log|L^*[\chi(t_n)]|^{-1}$ is asymptotically less than $r^{-1}\log\mathfrak{H}(r^2t_n)$ as $n\to\infty$. Again $m(t_n)$ asymptotically exceeds

$$t_n^{-2}\exp\{-d\exp[r\chi(t_n)]\},$$

so that $\log\log[1/m(t_n)]$ is asymptotically less than $\log\mathfrak{H}(r^2t_n)$ as $n\to\infty$. Using the relation $\log\mathfrak{H}(r^2t_n)$ is asymptotically less than $r^{3/2}\log\mathfrak{H}(r^{1/2}t_n)$ as $n\to\infty$, see [2; § 2], we obtain $|L_n(t_n)|\geqq\exp[-\mathfrak{H}(r^{1/2}t_n)]$ for n sufficiently large. Referring to (11) we see that if $\lambda_n(u)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{itu}\Lambda_n(t)dt$. then

$$|\lambda_n(u)|\leqq c'\exp[-|u|/rv(|u|)]\exp[+\mathfrak{H}(r^{1/2}t_n)].$$

Using assumption 1, we obtain (7) with $\mu=r^{1/2}$.

We can now complete our proof. Let $f(x)$ be any function in $L_2(-\infty,\infty)$. We associate with $f(x)$ the formal series $S(f)=\sum_{n=1}^{\infty}b_n(f)\Phi(t_n-x)$, where

$$b_n(f)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}\Psi_n(x)f(x)dx.$$

By (5) and Schwarz's inequality we see that for x fixed

$$|b_n(f)\Phi(t_n-x)|/\|f\|=O(\exp[-\mathfrak{H}(t_n-x)+\mathfrak{H}(\mu t_n)]),\quad n\to\infty.$$

Choose μ', $\max(\mu,r^2)<\mu'<1$. If n is sufficiently large $t_n-x\geqq\mu' t_n$, and hence $-\mathfrak{H}(t_n-x)+\mathfrak{H}(\mu t_n)\leqq-\mathfrak{H}(\mu' t_n)+\mathfrak{H}(\mu t_n)$. By [2; § 2] we have $\log\mathfrak{H}(\mu t_n)$ is asymptotically less than $(\mu/\mu')\log\mathfrak{H}(\mu' t_n)$, from which it follows that $-\mathfrak{H}(\mu' t_n)+\mathfrak{H}(\mu t_n)\sim-\mathfrak{H}(\mu' t_n)$ as $n\to\infty$. Finally $\mu' t_n\geqq H(n)$ for n sufficiently large. Combining our estimates we have $-\mathfrak{H}(t_n-x)+\mathfrak{H}(\mu t_n)$ is asymptotically less than $-n$ as $n\to\infty$. It follows that there exists a function $M(x)$ finite almost everywhere depending only on x and $\{t_n\}_1^{\infty}$ and $v(t)$, but independent of $f(x)$ such that $|S(f)|\leqq M(x)\|f\|$, $-\infty<x<\infty$. We now assume that $f(x)$ lies in the closure of the linear subspace of $L_2(-\infty,\infty)$ formed by the finite linear combinations of the functions $\{\Phi(t_n-x)\}_1^{\infty}$. There will then exist a sequence $\{F_k(x)\}_{k=1}^{\infty}$ of functions of the form

$$F_k(x)=\sum_{n=1}^{N_k}b_n^{(k)}\Phi(t_n-x)$$

such that $\lim \| f(x) - F_k(x) \| = 0$. It is no restriction to suppose that $\lim F_k(x) = f(x)$ almost everywhere since we can certainly select a subsequence with this property, and, if necessary, replace our original sequence by the subsequence. We have

$$| S(f) - S(F_k) | = | S(f - F_k) | \leqq \| f - F_k \| M(x).$$

But $S(F_k) = F_k$; thus $S(f) = \lim F_k = f(x)$ for almost all x. Q. E. D.

If we set $\nu \equiv 1$, the special case of our theorem which results may be applied to $e^{x/2} \exp[-e^x]$ to give Schwartz's theorem with the conditions

$$\sum_{n=1}^{\infty} e^{-t_n} < \infty, \liminf_{n \to \infty} [t_{n+1} - t_n] e^{t_n} > 0,$$

replaced by the more restrictive conditions

$$\liminf_{n \to \infty} t_n / \log n > 1, \liminf_{t = +\infty} [t_{n+1} - t_n] n > 0.$$

WASHINGTON UNIVERSITY.

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TAUBERIAN THEOREMS FOR SUMMABILITY (R_1).*

By Otto Szász.

1. Introduction. A series $\sum_0^\infty a_n$ is called summable (R_1) to the value s if the series

$$\sum_1^\infty s_n n^{-1} \sin nh, \qquad s_n = \sum_0^n a_\nu \tag{1.1}$$

converges in some interval $0 < h < h_0$, and if

$$2/\pi \sum_1^\infty s_n n^{-1} \sin nh \to s \text{ as } h \to 0. \tag{1.2}$$

It is convenient to assume $a_0 = 0$.

The method (R_1) is not regular, as the regularity condition $\sum_1^\infty n^{-1} |\sin nt| < \text{constant}$ is not satisfied.

If the series (1.1) converges in $0 < h < h_0$, then $s_n n^{-1} \sin nh \to 0$, $0 < h < h_0$, hence, by a well known theorem of G. Cantor,

$$s_n/n \to 0, \qquad n \to \infty. \tag{1.3}$$

If $s_n \equiv 1$, then $\sum_1^\infty n^{-1} \sin nh = (\pi - h)/2 \to \pi/2$, so that (1.2) holds.

A related method is summability $(R, 1)$, sometimes called Lebesgue's method. If the series $\sum_1^\infty a_n n^{-1} \sin nh$ is convergent in some interval $0 < h < h_0$, and if $\sum a_n (nh)^{-1} \sin nh \to s$, $h \to 0$, then $(R, 1) \sum a_n = s$. Several Tauberian theorems can be found in the literature, that is conditions under which summability $(R, 1)$ implies convergence and vice versa. We give here similar theorems for summability (R_1). Recently Hardy and Rogosinski [1] gave necessary and sufficient conditions for a function $f(t)$ in order that its Fourier series be summable (R_1). Our conditions are mainly on the terms a_n of a series $\sum a_n$.

2. Some lemmas.

Lemma 1. *If the series* (1.1) *converges in* $0 < h < h_0$, *then*

$$\sum_1^\infty s_n n^{-1} \sin nh = \sum_1^\infty a_n \rho_n(h), \tag{2.1}$$

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where

$$\rho_n(h) = \sum_{\nu=n}^{\infty} \nu^{-1} \sin \nu h. \tag{2.2}$$

Conversely, if $n^{-1}s_n \to 0$, then the convergence of the right hand side implies (2.1).

Partial summation yields

$$\sum_{n}^{m} s_\nu - 1 \sin \nu h = s_n\rho_n - s_m\rho_{m+1} + \sum_{n+1}^{m} a_\nu\rho_\nu, \tag{2.3}$$

and $\rho_n(h) = -(\tau_{n-1})/n + \sum_{n}^{\infty} \tau_\nu/[\nu(\nu+1)]$, where

$$\tau_n(h) = \sum_{1}^{n} \sin \nu h = (\sin nh)/2 + \cot(h/2)\sin^2(nh/2), \qquad 0 < h < \pi.$$

It follows that $|\rho_n(h)| < n^{-1}|\tau_{n-1}| + \sum_{n}^{\infty} [\nu(\nu+1)]^{-1}|\tau_\nu|$. Now $|\tau_n(h)| < 1/2 + [\sin(h/2)]^{-1}$, hence

$$|\rho_n(h)| < n^{-1}[1 + 2/\sin(h/2)], \tag{2.4}$$

and, in view of (1.3), $s_n\rho_{n+1} \to 0$, for fixed h, as $n \to \infty$. Thus $\sum a_\nu\rho_\nu$ converges, and

$$\sum_{n}^{\infty} s_\nu \nu^{-1} \sin \nu h = \sum_{n+1}^{\infty} a_\nu\rho_\nu(h) + s_n\rho_n(h); \tag{2.5}$$

$n = 1$ yields (2.1). The converse follows similarly from (2.3) and (2.4).

We shall call a series $\sum a_n$ summable $(\bar{R}_1)$ to s if $\sum_{1}^{\infty} a_n\rho_n(h)$ converges for small positive h, and if $2/\pi \sum_{1}^{\infty} a_n\rho_n(h) \to s$, $h \to 0$. It follows from Lemma 1 that summability $(\bar{R}_1)$ and $n^{-1}s_n \to 0$ imply summability (R_1).

LEMMA 2. *Suppose that* $0 < \delta \leq 1$,

$$\sum_{n}^{2n} (|a_\nu| - a_\nu) = O(n^{1-\delta}), \tag{2.6}$$

and

$$s_n = \sum_{0}^{n} a_\nu = O(n^{1-\delta}); \tag{2.7}$$

then $\sum_{1}^{\infty} \nu^{-1}|a_\nu|$ *converges, and* (see [3], p. 579)

$$\sum_{n}^{\infty} \nu^{-1}|a_\nu| = O(n^{-\delta}). \tag{2.8}$$

LEMMA 3. *If* $\sum a_n$ *is Abel-summable, and if*

$$\sum_{n}^{2n} (|a_\nu| - a_\nu) = O(1), \tag{2.9}$$

then $\sum a_n$ *is summable* $(C, 1)$ *(Cesàro mean of order* 1*), and*

$$s_n = O(1). \tag{2.10}$$

See [4], Lemmas 1 and 3.

3. Tauberian conditions for summability (R_1).

THEOREM 1. *If* (2.9) *holds, then Abel-summability of* $\sum a_n$ *implies summability* (R_1) *to the same value.*

Let $\sum_1^n s_\nu = s'_n$; by Lemma 3, $n^{-1}s'_n$ tend to a limit s; we may assume without loss of generality that $s = 0$ (otherwise replace a_1 by $a_1 - s$).

We write

$$\sum_1^\infty s_\nu \nu^{-1} \sin \nu h = \sum_1^n + \sum_{n+1}^\infty = A_n(h) + B_n(h), \text{ say.} \tag{3.1}$$

Summation by parts gives

$$A_n(h) = s'_n n^{-1} \sin nh + \sum_1^{n-1} s'_\nu [(\sin \nu h)/\nu - \sin(\nu+1)h/(\nu+1)].$$

But

$$\nu^{-1} \sin \nu h - \sin(\nu+1)h/(\nu+1)$$
$$= \sin(\nu+1)h/\nu(\nu+1) - (2/\nu)\sin(h/2)\cos(2\nu+1)h/2,$$

hence

$$|(\sin \nu h)/\nu - \sin(\nu+1)h/(\nu+1)| < h/\nu + h/\nu = 2h/\nu.$$

Given a positive $\epsilon < 1/2$, choose $n_0(\epsilon)$ so that $|s'_n| < \epsilon^2 n/2$ for $n > n_0$; then

$$\sum_1^n \nu^{-1} |s'_\nu| < \epsilon^2 n, \text{ for } n > n_1(\epsilon) > n_0.$$

To any h in the interval $0 < h < 1/n_1$ choose $n = 1 + (\epsilon h)^{-1}$, so that

$$n > (\epsilon h)^{-1} > n_1/\epsilon > 2n_1, \text{ and } n < 1 + (\epsilon h)^{-1}. \tag{3.2}$$

Now

$$|A_n(h)| < \epsilon^2 + 2h \sum_1^n \nu^{-1} |s'_\nu| < \epsilon^2(1 + 2hn)$$
$$< \epsilon^2(1 + 2h + 2/\epsilon) < 2\epsilon + 3\epsilon^2.$$

Next, using (2.3) and (2.4) we get convergence of the series (3.1), and from (2.4) and (2.5)

$$(3.3)\quad B_n(h) < |s_{n+1}|(n+1)^{-1}[1+2/\sin(h/2)] + [1+2/\sin(h/2)]\sum_n^\infty |a_\nu|/\nu.$$

From (2.10) and (3.2), $s_{n+1}(n+1)^{-1}[1+2/\sin(h/2)] = O(\epsilon)$, and from (2.8) for $\delta = 1$,

$$(3.4)\qquad \sum_n^\infty \nu^{-1}|a_\nu| = O(1/n).$$

It follows that $\sum_1^\infty s_\nu(\sin \nu h)/\nu = O(\epsilon)$ for $h < h_0(\epsilon)$. This proves Theorem 1.

The analogous theorem holds for summability $(R, 1)$; see [4,], Theorem 4. There is also given a Fourier power series, satisfying the assumptions, while $\sum a_n$ diverges.

THEOREM 2. *If* (2.9) *holds and if* $\sum a_n$ *converges, then* (1.1) *converges uniformly in* $0 \le h \le \pi$. *In particular* $\sum a_n$ *is summable* (R_1).

We may assume that $s_n \to 0$; we shall show that $B_n(h)$ is uniformly small for large n. We write

$$B_n(h) = \sum_{n+1}^{\lambda n} + \sum_{\nu > \lambda n}, \text{ where } \lambda > 1.$$

From (3.3) and (3.4), $\sum_{\nu>\lambda n} s_\nu \nu^{-1} \sin \nu h = O(\lambda n h)^{-1}$. Given $\epsilon > 0$, choose $n_0(\epsilon)$ so that $|s_n| < \epsilon^2$ for $n > n_0$; define $k = k(\lambda)$ by $n + k = [\lambda n]$, so that $k \le (\lambda - 1)n$. To any h in $0 < h < \pi$ choose $\lambda = 1 + (\epsilon n h)^{-1}$. Then

$$\left|\sum_{n+1}^{\lambda n} s_\nu \nu^{-1} \sin \nu h\right| < h \sum_{n+1}^{\lambda n} |s_\nu| < hk\epsilon^2(\lambda - 1)n = \epsilon,$$

and $\lambda n h = nh + 1/\epsilon > 1/\epsilon$, hence $B_n(h) = O(\epsilon)$, for $n > n_0(\epsilon)$. This proves Theorem 2. The analogous theorem holds for summability $(R. 1)$; see [4], Theorem 5.

Note that the convergence of $\sum a_n$ is not a necessary condition for the uniform convergence of (1.1) in the interval $(0, \pi)$. This is seen from the example in which s_n is 1 or 0 according as n is or is not a power of 2.

4. Summability (R_1) of Fourier series. The following theorem is due to Hardy and Rogosinski [1, Theorem 3]:

In order that the Fourier series

$$(4.1)\qquad f(t) \sim \sum_1^\infty a_n \cos nt$$

should be summable (R_1) to 0 for $t=0$, it is necessary and sufficient that the Cauchy integral

$$I(h)=\int_{\to 0}^{\pi} t^{-1}dt\int_{|h-t|}^{h+t} f(u)\cot(u/2)\,du$$

should converge for small h and tend to 0 with h. The upper limit may be replaced by any positive $\delta<\pi$. Hardy and Rogosinski have constructed a class of Fourier series which are nowhere summable (R_1). It is all the more remarkable that a Fourier series is summable (R_1) at each point of continuity.

For the proof we may assume that $t=0$ is a point of continuity and that $f(0)=0$. We write

$$I(h)=\int_{\to 0}^{h}+\int_{h}^{h^{\frac{1}{2}}}+\int_{h^{\frac{1}{2}}}^{\pi}=I_1+I_2+I_3,$$

say. Now

$$I_1(h)=o\left(\int_{\to 0}^{h}t^{-1}dt\int_{h-t}^{h+t}u^{-1}du\right)=o\left(\int_{\to 0}^{h}t^{-1}\log(h+t)/(h-t)\,dt\right);$$

the transformation $(h+t)/(h-t)=1+x$, $t=hx/(2+x)$, yields

$$I_1(h)=o\left(\int_0^{\infty}x(2+x)^{-1}\log(1+x)\,dx\right)=o(1). \tag{4.2}$$

Next

$$I_2(h)=o\int_{h}^{h^{\frac{1}{2}}}t^{-1}dt\int_{t-h}^{t+h}u^{-1}du=o\int_{h}^{h^{\frac{1}{2}}}t^{-1}\log(t+h)/(t-h)\,dt,$$

and the transformation $(t+h)/(t-h)=1+x$, $t=h(2+x)/x$, yields

$$I_2(h)=o\int_{2h/h^{\frac{1}{2}}-h}^{\infty}[x(2+x)]^{-1}\log(1+x)\,dx=o(1). \tag{4.3}$$

Finally

$$\begin{aligned}I_3(h)&=O\int_{h^{\frac{1}{2}}}^{\pi}t^{-1}dt\int_{t-h}^{t+h}u^{-1}f(u)\,du\\&=O\int_{h^{\frac{1}{2}}}^{\pi}[t(t-h)]^{-1}[f_1(t+h)-f_1(t-h)]dt,\end{aligned}$$

if $f_1(t)=\int_0^t|f(u)|\,du$.

Now $\int_{h^{\frac{1}{2}}}^{\pi}f_1(t-h)/[t(t-h)]dt=\int_{h^{\frac{1}{2}}-2h}^{\pi-2h}f_1(t+h)/[(t+2h)(t+h)]dt,$

so that

$$I_3(h) = O\left(\int_{h^{\frac{1}{3}}-2h}^{\pi-2h} f_1(t+h)\{[t(t-h)]^{-1} - [(t+2h)(t+h)]^{-1}\}dt\right.$$

$$\left. + \int_{\pi-2h}^{\pi} f_1(t+h)/[t(t-h)]dt\right). \quad \text{Furthermore} \int_{\pi-2h}^{\pi} f_1(t+h)/[t(t-h)]dt$$

$$= O\int_{\pi-2h}^{\pi}(t-h)^{-1}dt = O(h) = o(1), \text{ and}$$

$$\int_{h^{\frac{1}{3}}-2h}^{\pi-2h} f_1(t+h)\{[t(t-h)]^{-1} - [(t+2h)(t+h)]^{-1}\}dt$$

$$= 2h\int_{h^{\frac{1}{3}}-h}^{\pi-h} f_1(t)(2t-h)/[t(t-h)(t+h)(t-2h)]dt$$

$$= O\left(h\int_{h^{\frac{1}{3}}-h}^{\pi-h}(2t-h)/[(t-h)(t+h)(t-2h)]dt\right)$$

$$= O\left(\int_{h^{\frac{1}{3}}}^{\pi}(2t-3h)h/[(t-2h)t(t-3h)]dt\right)$$

$$= O\left(\int_{h^{\frac{1}{3}}}^{\pi} h/(t-3h)^2 dt\right)$$

$= O(h/(h^{\frac{1}{3}}-3h)) = o(1)$. Thus $I_3(h) = o(1)$, $h \to 0$. Hence, the estimates (4. 2)-(4. 3) prove the following theorem.

THEOREM 3. *The Fourier series of a function $f(t)$ is summable (R_1) at each point of continuity of $f(t)$.*

5. The Lebesgue constants. Employing well known results, Theorem 3 also follows if we prove that the set of Lebesgue constants corresponding to the transform (R_1) is bounded. We assume that $f(t)$ is integrable and that $|f(t)| \leq 1$, $0 \leq t \leq \pi$. Let

$$s_n = s_n(\theta) = \sum_1^n a_\nu \cos \nu\theta = (1/2)a_n \cos n\theta + s_n^*(\theta).$$

Then $s_n^*(\theta) = 1/\pi \int_0^\pi \phi(t)\cos(t/2)\sin nt\, dt$, where

$$\phi(t) = (1/2)[f(\theta+t) + f(\theta-t)].$$

Because of periodicity, we may restrict ourselves, as usual to $\theta = 0$. Now

$$\sum_1^\infty s_n n^{-1} \sin nh = (1/2)\sum_1^\infty a_n n^{-1} \sin nh + \sum_1^\infty s_n^* n^{-1} \sin nh;$$

here $|\sum_1^\infty a_n n^{-1} \sin nh| = \left|\int_0^h f(t)dt\right| \leq h/2 \leq \pi/2$ for $0 < h < \pi$. Next, from [1], § 3. 1,

$$\sum_1^\infty s_n^* n^{-1} \sin nh = 1/\pi \int_0^\pi f(t)\cot(t/2)\sum_1^\infty n^{-1} \sin nt \sin nh\, dt,$$

where, if $0 < t < \pi$, $0 < h < \pi$,

$$\sum_1^\infty n^{-1} \sin nt \sin nh = (1/2)\log[\sin(1/2)(t+h)]/[\sin(1/2)|t-h|] > 0.$$

It now follows that $|\sum_1^\infty s_n^* n^{-1} \sin nh|$

$$\leq (1/2\pi)\int_0^\pi \cot(t/2)\log[\sin(1/2)(t+h)]/[\sin(1/2)|t-h|]dt$$

$$= (1/\pi)\int_0^\pi \cot(t/2)\sum_1^\infty n^{-1}\sin nt \sin nh\, dt$$

$$= (1/\pi)\sum_1^\infty n^{-1}\sin nh \int_0^\pi \cot(t/2)\sin nt\, dt = \sum_1^\infty n^{-1}\sin nh$$

$= (\pi - h)/2 \leq \pi$, termwise integration being legitimate, as in a similar case in [1], p. 178.

This proves that the Lebesgue constants are uniformly bounded.

6. Other conditions for summability (R_1).

THEOREM 4. *If*

$$s_n - s = o(1/\log n), \tag{6.1}$$

and if for some $\delta > 0$

$$\sum_n^{2n}(|a_\nu| - a) = O(n^{1-\delta}), \tag{6.2}$$

then $\sum a_n$ *is summable* (R_1).

Summability $(R, 1)$ under the same assumptions was proved in [3], p. 578.

We may assume that $s = 0$ and $\delta < 1$; we write

$$\sum_1^\infty s_\nu \nu^{-1} \sin \nu h = \sum_1^n + \sum_{n+1}^\infty = C_n(h) + D_n(h), \tag{6.3}$$

say. It follows from Lemma 2 that $\sum \nu^{-1}|a_\nu| < \infty$; from (6.1) and (2.4) for fixed h, $s_n \rho_{n+1} \to 0$, and from (2.3) follows the convergence of (6.3).

We choose $n = (1/h)$, $o < h < 1/2$, then, in view of (6.1),

$$C_n(h) = O(h \sum_{\nu h \leq 1} |s_\nu|) = O[1/\log(1/h)] = o(1), \qquad h \to 0. \tag{6.4}$$

We next choose an r such that $r\delta > 1$; to any $h < 1/4$ choose $k_n = [h^{-r}] \leq h^{-r}$. Now

$$\sum_{n+1}^{k_n} s_\nu \nu^{-1} \sin \nu h = o(\sum_{n+1}^{k_n} 1/\nu \log \nu) = o(\log\log k_n - \log\log n) \tag{6.5}$$

$$= o(\log(\log k_n)/(\log n)) = o(\log r \log(1/h)/\log(1/h))$$

$= \log r \cdot o(1)$. Finally, employing Lemma 2 and formulas (2.4), (2.5), we have

$$(6.6) \qquad \sum_{k_n}^{\infty} s_\nu \nu^{-1} \sin \nu h = h^{-1} O(k_n^{-\delta}) = h^{-1} O(h^{r\delta}) = O(h^{r\delta - 1}) = o(1).$$

Combining (6.3)-(6.6) yields Theorem 4.

7. A $(C, r) \to (R_1)$ theorem.

THEOREM 5. *If $\sum a_n$ is summable $(C, 1-\alpha)$ for some positive $\alpha < 1$ and if*

$$(7.1) \qquad \sigma_n \equiv \sum_{1}^{n} | s_\nu^{-\alpha} | = O(n^{1-\alpha}),$$

then the series is summable (R_1).

The corresponding theorem holds for summability $(R, 1)$; see [5], Theorem A. For closely related theorems see [2].

The Cesàro sums s_n^r are defined by

$$(7.2) \qquad s_n^r = \sum_{1}^{n} \gamma_{n-\nu}^r a_\nu, \text{ where } \gamma_n^r = \binom{n+r}{n}.$$

We have $\gamma_n^r - \gamma_{n-1}^r = \gamma_n^{r-1}$,

$$(7.3) \qquad s_n^r = \sum_{1}^{n} \gamma_{n-\nu}^{\epsilon-1} s_\nu^{r-\epsilon}, \qquad s_n^0 = s_n,$$

$$(7.4) \qquad \gamma_n^r \sim n^r/\Gamma(r+1), \qquad n \to \infty, r \neq -1, -2, \cdots.$$

By asumption, $s_n^{1-\alpha}/\gamma_n^{1-\alpha} \to s$, $n \to \infty$; we may assume $s = 0$, so that, in view of (7.4)

$$(7.5) \qquad n^{\alpha-1} s_n^{1-\alpha} \to 0, \qquad n \to \infty.$$

From (7.3), for $r \equiv 0$ and $\epsilon = \alpha - 1$, we have $s_n = \sum_{1}^{n} \gamma_{n-\nu}^{\alpha-2} s_\nu^{1-\alpha}$; hence, from (7.4) and (7.5), $s_n = O(n^{1-\alpha}) = o(n)$. Thus, by Lemma 1, we need only to prove summability $(\bar{R}_1)$ to zero.

We have $a_n = \sum_{1}^{n} \gamma_{n-\nu}^{\alpha-2} s_\nu^{-\alpha}$, hence

$$(7.6) \qquad \sum_{1}^{\infty} a_n \rho_n(h) = \sum_{1}^{\infty} \rho_n(h) \sum_{\nu=1}^{n} \gamma_{n-\nu}^{\alpha-2} s_\nu^{-\alpha} = \sum_{\nu=1}^{\infty} s_\nu^{-\alpha} \sum_{n=\nu}^{\infty} \gamma_{n-\nu}^{\alpha-2} \rho_n(h);$$

the interchange of summation is justified if the double series is absolutely convergent. Now, from (2.4) and (7.4),

$$\sum_{n \geq \nu} | \gamma_{n-\nu}^{\alpha-2} \rho_n(h) | = O \sum_{n \geq \nu} n^{-1} (n - \nu + 1)^{-2}$$

$= O(\nu^{-1})$ for $\alpha < 1$. Furthermore

$$\sum_1^m \nu^{-1} \mid s_\nu^{-\alpha} \mid = m^{-1}\sigma_m + \sum_1^{m-1} \sigma_\nu/\nu(\nu+1) = O(m^{-\alpha}) + O\sum_1^m \nu^{1-\alpha}/\nu(\nu+1)$$

$= O(1)$ as $0 < \alpha < 1$; this proves (7.6).

Summation by parts gives

$$\sum_{n=\nu}^m \gamma_{n-\nu}{}^{\alpha-2}\rho_n(h) = \gamma_{m-\nu}{}^{\alpha-1}\rho_m(h) + \sum_{n=\nu}^{m-1} \gamma_{n-\nu}{}^{\alpha-1}(\rho_n - \rho_{n+1})$$

$= O(m^{\alpha-2}) + \sum_{n=\nu}^{m-1} \gamma_{n-\nu}{}^{\alpha-1} n^{-1} \sin nh$, $\alpha < 2$, hence

$$\sum_{n=\nu}^\infty \gamma_{n-\nu}{}^{\alpha-2}\rho_n(h) = \sum_{n=\nu}^\infty \gamma_{n-\nu}{}^{\alpha-1} n^{-1} \sin nh \equiv \Delta^{-\alpha}\nu^{-1} \sin \nu h.$$

Now, from (7.6)

$$(7.7) \qquad \sum_1^\infty a_n\rho_n(h) = \sum_{\nu=1}^\infty s_\nu{}^{-\alpha}\Delta^{-\alpha}\nu^{-1} \sin \nu h = \sum_{\nu h \le \lambda} + \sum_{\nu h > \lambda}$$

$= \phi_1(h) + \phi_2(h)$, say, where $\lambda > 1$ is fixed.

We wish to show that $\phi_1(h) \to 0$ and $\phi_2(h) \to 0$, as $h \to 0$. To this end we first estimate $\Delta^r \nu^{-1} \sin \nu h$ for $r > -1$. By definition,

$$\Delta^r \nu^{-1} \sin \nu h = \sum_{n=\nu}^\infty \gamma_{n-\nu}{}^{-r-1} n^{-1} \sin nh, \qquad r > -1.$$

Now $n^{-1} \sin nh = h/2 \int_{-1}^1 e^{inhx} dx$, hence

$$\Delta^r \nu^{-1} \sin \nu h = (h/2) \sum_\nu^\infty \gamma_{n-\nu}{}^{-r-1} \int_{-1}^1 e^{inhx} dx = h/2 \int_{-1}^1 \sum_\nu^\infty \gamma_{n-\nu}{}^{-r-1} e^{inhx} dx,$$

interchange of summation and integration being legitimate. But

$$\sum_{n=\nu}^\infty \gamma_{n-\nu}{}^{-r-1} e^{inhx} = \sum_{k=0}^\infty k^{-r-1} e^{i(\nu+k)hx}$$

$= e^{i\nu hx}(1 - e^{ihx})^{-r}$, hence

$$\Delta^r \nu^{-1} \sin \nu h = h/2 \int_{-1}^1 e^{i\nu hx}(1 - e^{ihx})^{-r} dx$$

It follows that

$$(7.8) \qquad \mid \Delta^r \nu^{-1} \sin \nu h \mid < h \int_{-1}^1 \mid 1 - e^{ihx} \mid^{-r} dx = O(h^{r+1}).$$

We now assume $0 < h < 1/2$, and write

$$(7.9) \qquad [\lambda h^{-1}] = n;$$

thus

$$\phi_1(h) = \sum_{\nu=1}^{n} s_\nu{}^{-\alpha}\Delta^{-\alpha}\nu^{-1}\sin \nu h = s_n{}^{1-\alpha}\Delta^{-\alpha}n^{-1}\sin nh + \sum_{1}^{n-1} s_\nu{}^{1-\alpha}\Delta^{1-\alpha}\nu^{-1}\sin \nu h.$$

Employing (7. 5) and (7. 8) we get

$$\phi_1(h) = o(n^{1-\alpha}h^{1-\alpha}) + o(\sum_{1}^{n-1}\nu^{1-\alpha}h^{2-\alpha})$$

$= o(nh)^{1-\alpha} + o(nh)^{2-\alpha}$, and in view of (7. 9)

$$\phi_1(h) = o(\lambda^{1-\alpha}) + o(\lambda^{2-\alpha}) = o(\lambda^{2-\alpha}). \tag{7. 10}$$

To estimate $\phi_2(h)$, consider again $\Delta^{-\alpha}\nu^{-1}\sin \nu h$, and write

$$\Delta^{-\alpha}\nu^{-1}\sin \nu h = \sum_{k=\nu}^{\nu+\eta}\gamma_{k-\nu}{}^{\alpha-1}k^{-1}\sin kh + \sum_{k=\nu+\eta+1}^{\infty}\gamma_{k-\nu}{}^{\alpha-1}k^{-1}\sin kh$$

$= \delta_1(h) + \delta_2(h)$, say, where $\eta = [h^{-1}]$. Now

$$\delta_1(h) = O\sum_{k=\nu}^{\nu+\eta}(k-\nu+1)^{\alpha-1}k^{-1} = O(\nu^{-1}\eta^\alpha) = O(\nu^{-1}h^{-\alpha}),$$

and, summing by parts,

$$\begin{aligned}\delta_2(h) &= -\gamma_{\eta+1}{}^{\alpha-1}\rho_{\nu+\eta}(h) + \sum_{k=\nu+\eta+1}^{\infty}\gamma_{k-\nu}{}^{\alpha-2}\rho_k(h)\\ &= O(\eta^{\alpha-1}[h(\nu+\eta)]^{-1}) + O(\sum_{k=\nu+\eta+1}^{\infty}(k-\nu+1)^{\alpha-2}(kh)^{-1})\\ &= O(\eta^{\alpha-1}[h(\nu+\eta)]^{-1}) + O(\eta^{\alpha-1}[h(\nu+\eta)]^{-1})\end{aligned}$$

$= O(\eta^{\alpha-1}(\nu h)^{-1}) = O(\nu^{-1}h^{-\alpha})$. Thus $\Delta^{-\alpha}\nu^{-1}\sin \nu h = O(\nu^{-1}h^{-\alpha})$. Now

$$\phi_2(h) = O(\sum_{\nu h > \lambda} | s_\nu{}^{-\alpha} | \nu^{-1}h^{-\alpha}),$$

furthermore, employing (7. 1),

$$\sum_{k}^{\infty} | s_\nu{}^{-\alpha} | \nu^{-1} = k^{-1}\sigma_{k-1} + \sum_{k}^{\infty}\sigma_\nu/[\nu(\nu+1)]$$

$= O(\sum_{k}^{\infty}\nu^{-1-\alpha}) = O(k^{-\alpha})$, hence

$$\phi_2(h) = O(h^{-\alpha}(\lambda/h)^{-\alpha}) = O(\lambda^{-\alpha}). \tag{7. 11}$$

Combining (7. 7), (7. 10) and (7. 11), we get $\sum_{n=1}^{\infty} a_n\rho_n(h) = o(\lambda^{2-\alpha} + O(\lambda^{-\alpha})$; thus, letting $h \to 0$,

$$\limsup_{h\to 0} | \sum_{1}^{\infty} a_n\rho_n(h) | \leq O(\lambda^{-\alpha}).$$

Here α is positive, $O(1)$ is independent of λ, and λ is arbitrarily large, hence $\sum a_n\rho_n(h) \to 0$ as $h \to 0$. This proves Theorem 5.

8. Connection with summability $(k, 1)$. The well known formula

$$\sum_1^n \nu^{-1} \sin \nu t = -(t/2) + \int_0^t \{\sin[n+(1/2)]x/[2\sin(x/2)]\}dx$$

yields

$$\sum_1^{n-1} \nu^{-1} \sin \nu t + (1/2)n^{-1} \sin nt = -(t/2) + (1/2)\int_0^t \sin nx \cot(x/2)\,dx,$$

hence, for $0 < t < \pi$,

$$\rho_n(t) = \pi/2 + (1/2)n^{-1} \sin nt - (1/2)\int_0^t \sin nx \cot(x/2)\,dx.$$

Furthermore $\int_0^t x^{-1} \sin nx dx = \int_0^{nt} y^{-1} \sin y dy = \pi/2 - \int_{nt}^\infty y^{-1} \sin y dy$, so that

$$\rho_n(t) = \int_{nt}^\infty y^{-1} \sin y dy + (1/2)n^{-1} \sin nt + \int_0^t [x^{-1} - (1/2)\cot(x/2)] \sin nx dx$$

$$= Si(nt) + (1/2)n^{-1} \sin nt + \int_0^{t/2} (y \sin y)^{-1}(\sin y - y \cos y) \sin 2ny dy.$$

The formula $(d/dy)(y^{-1} - \cot y) = (\sin^2 y)^{-1} - y^{-2}$ and an integration by parts yield

$$\rho_n(t) = Si(nt) + (1/2)n^{-1} \sin nt + (t^{-1} - (1/2)\cot(t/2))n^{-1}(1 - \cos nt)$$

$$-(1/2)\int_0^{t/2} [(\sin^2 y)^{-1} - y^{-2}]n^{-1}(1 - \cos 2ny)\,dy.$$

We now have

$$(8.1) \quad \sum_1^n a_\nu \rho_\nu(t) = \sum_1^n a_\nu Si(\nu t) + (1/2)\sum_1^n \nu^{-1} a_\nu \sin \nu t$$

$$+ ((t^{-1} - (1/2)\cot(t/2))\sum \nu^{-1} a_\nu (1 - \cos \nu t)$$

$$-(1/2)\int_0^{t/2} ((\sin^2 y)^{-1} - y^{-2})\sum_1^n \nu^{-1} a_\nu (1 - \cos 2\nu y) dy.$$

We now suppose that both (4.1) and its conjugate

$$(8.2) \quad \sum_1^\infty a_n \sin nt \sim \tilde{f}(t)$$

are Fourier series. It then follows from [1], Theorem 7, that $\sum a_n$ is summable (R_1) if and only if the Cauchy integral

$$(8.3) \quad \int_{\to 0}^{\pi} \tilde{f}(t)/t\,dt$$

exists. We give here a simpler proof, based on formula (8.1). We now have

$$\sum_1^\infty n^{-1}a_n \sin nt = \int_0^t f(x)\,dx, \qquad \sum_1^\infty n^{-1}a_n(1-\cos nt) = \int_0^t \tilde{f}(x)\,dx,$$

and, from (8.1),

$$\sum_1^\infty a_\nu\{\rho_\nu(t) - Si(\nu t)\} = 1/2 \int_0^t f(x)\,dx + (t^{-1} - (1/2)\cot(t/2)) \int_0^t \tilde{f}(x)\,dx$$
$$- 1/2 \int_0^{t/2} \{(\sin^2 y)^{-1} - y^{-2}\} \int_0^{2y} f(x)\,dx\,dy.$$

Here each term on the right tends to 0, as $t \to 0$. Thus the series $\sum_1^\infty a_\nu \rho_\nu(t)$ and $\sum a_\nu Si(\nu t)$ are either both convergent or both divergent. Now from (8.2)

$$\sum_1^\infty a_\nu Si(\nu t) = \int_t^\infty \tilde{f}(x)/x\,dx, \qquad t > 0,$$

and the series converges for all $t > 0$; hence $\sum a_\nu \rho_\nu(t)$ is convergent for all $t > 0$. Furthermore $s_n = o(n)$, hence by Lemma 1, the series (1.1) is convergent. It is now clear that summability (R_1) implies (8.3) and conversely. We call the series $\sum a_n$ summable $(k, 1)$ to s if the series $\sum a_n Si(nt)$ converges in some interval $0 < t < t_0$, and if $2/\pi \sum a_n Si(nt) \to s$, as $t \to 0$. We have just shown that under the assumptions (4.1) and (8.2) the summability methods (R_1) and $(k, 1)$ are equivalent.

9. Gibbs phenomenon. In addition to the regularity of the corresponding Lebesgue constants, the transform (R_1) presents no Gibbs phenomenon. Consider the transform of the series $\sum_1^\infty n^{-1} \sin nt$, so that

$$s_n = -(t/2) + (1/2)n^{-1} \sin nt + (1/2) \int_0^t \cot(x/2) \sin nx\,dx;$$

thus the transform is

$$2/\pi \sum_1^\infty n^{-1}s_n \sin nh = -(t/\pi) \sum_1^\infty n^{-1} \sin nh + 1/\pi \sum_1^\infty n^{-2} \sin nt \sin nh$$
$$+ 1/\pi \int_0^t \cot(x/2) \left(\sum_1^\infty n^{-1} \sin nx \sin nh\right) dx.$$

It is easy to see that the transform sums the series to its proper value $(\pi - t)/2$ for $0 < t < \pi$. We wish to determine

$$\limsup_{t \to 0, h \to 0} 4/\pi^2 \sum n^{-1}s_n(t) \sin nh = \lambda,$$

which is called the Gibbs ratio of the transform (R_1); it is clear that $\lambda \geq 1$.

If $\lambda = 1$ then we say that the transform presents no Gibbs phenomenon. The first term on the right of (9.1) evidently tends to zero. The second term is in absolute value less than

$$1/2\pi \sum_1^\infty n^{-2}(\sin^2 nt + \sin^2 nh) = 1/2\pi\{t(\pi - t)/2 + h(\pi - h)/2\},$$

hence tends to zero. Finally the last term is positive and less than (see section 5)

$$1/\pi \int_0^\pi \cot)x/2)\,(\sum_1^\infty n^{-1} \sin nx \sin nh)\,dx = (\pi - h)/2 < \pi/2,$$

which proves our assertion.

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UNIVERSITY OF CINCINNATI

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LINEAR DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC OR LAPLACE TRANSFORM COEFFICIENTS.*

By CALVIN R. PUTNAM and AUREL WINTNER.

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1. This paper consists of two parts, dealing with an n-th order linear differential equation possessing coefficients all of which either are uniformly almost periodic functions or are Laplace transforms. The proof, given in part II, of the theorem dealing with Laplace transforms will be somewhat abbreviated owing to its similarity with the proof, which will be given in detail in part I, of the theorem in the almost periodic case.

PART I.

2. Let

$$f_0(t), f_1(t), \cdots, f_{n-1}(t), \qquad (-\infty < t < \infty), \tag{1}$$

be n uniformly almost periodic functions of the real variable t, having the property that the frequencies $\lambda_1, \lambda_2, \cdots$ occurring in the Fourier expansion,

$$\sum_k a_{kj} e^{-i\lambda_k t}, \qquad j = 0, 1, \cdots, n-1, \tag{2}$$

of any of the functions (1) have a positive lower bound; so that, without loss of generality,

$$\lambda_k \geqq 1, \tag{3}$$

if the unit of length on the t-axis is suitably chosen. (Since the functions $f_k(t)$ are considered on the entire t-axis, the choice of the minus sign in (2) is, at present, obviously immaterial; it is inserted, however, for the sake of later convenience.)

THEOREM I. The homogeneous, linear differential equation

$$\sum_{h=0}^{n} f_h(t)\, d^h x/dt^h = 0, \tag{4}$$

where $-\infty < t < \infty$ and the coefficients (1) are uniformly almost periodic on $-\infty < t < \infty$ with Fourier expansions (2) satisfying (3) and

$$f_n(t) \equiv 1, \tag{5}$$

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must have a non-trivial ($\not\equiv 0$) solution $x = x_1(t)$ which is bounded, as $t \to \pm\infty$, and is, to a constant factor, the only bounded solution of (4);

this solution is uniformly almost periodic;

if c denotes the constant term of the Fourier expansion of $x_1(t)$, then $c \neq 0$ (so that, since $x_1(t)$ can be multiplied by any constant, c can be assigned as an integration constant determining $x_1(t)$, the trivial solution $x(t) \equiv 0$ being determined by $c = 0$);

if the constant term of the Fourier expansion of $x_1(t)$ is subtracted from $x_1(t)$, then the resulting uniformly almost periodic function, $x_1(t) - c$, has frequencies all of which are linear combinations, with positive integral coefficients, of the frequencies of the coefficient functions (1) (so that, in particular, all non-vanishing frequencies of $x_1(t)$ must satisfy the inequality (3), the latter being assumed of all frequencies of the functions (1));

the frequency 0 of $x_1(t)$, represented by the occurrence of the term $c \neq 0$ in the Fourier expansion of $x_1(t)$, can be thought of as being due to the presence of the coefficient function (5) (which is uniformly almost periodic with 0 as its only frequency);

finally, the general solution of (4) can be described as follows:

(4) has a system of n linearly independent solutions $x = x^1(t), \cdots, x^n(t)$, which can be chosen so as to possess the recursive structure

$$(6) \qquad x^1(t) = x_1(t),\ x^2(t) = tx^1(t) + x_2(t), \cdots, x^n(t) = tx^{n-1}(t) + x_n(t),$$

that is,

$$(6\text{ bis}) \qquad x^k(t) = \sum_{h=0}^{k-1} t^h x_{k-h}(t), \qquad k = 1, 2, \cdots, n,$$

where $x_1(t)$ is the uniformly almost periodic function singled out above, and the remaining $n-1$ functions $x_2(t), \cdots, x_n(t)$ have the structure

$$(7) \qquad x_k(t) = \sum_{h=0}^{k-2} t^h x_{kh}(t), \qquad k = 2, 3, \cdots, n,$$

where the functions x_{kh} are uniformly almost periodic with Fourier expansions satisfying (2) and (3) (and with frequencies all of which are linear combinations, with positive integral coefficients, of the frequencies of the coefficient functions (1)).

Corresponding results hold when (4) is replaced by the inhomogeneous differential equation

$$(8) \qquad \sum_{h=0}^{n} f_h(t)\, d^h x/dt^h = g(t),$$

if $g(t)$ is uniformly almost periodic and satisfies (3), as do the coefficient functions (1). It will be clear from the results of Section 5 below that (8) has a particular solution which is almost periodic and has a Fourier expansion (2) satisfying (3) with frequencies which are linear combinations, with positive integral coefficients, of the frequencies of the coefficient functions (1) and of $g(t)$. The general solution of (8) is obtained, of course, by adding to this particular solution the general solution of the linear homogeneous differential equation (4), as furnished by (6).

The truth of the assertion of the part of the theorem as to the frequencies of certain functions being linear combinations, with positive integral coefficients, of the frequencies of the coefficient functions, will be clear from the proof.

All of this parallels the results of [4], where $n = 2$, in the same way as [2] parallels [3].

In order to prove Theorem I, it will be convenient to transcribe the statement and terminology of this theorem in such a manner that the real line $-\infty < t < \infty$ is replaced by the complex closed half-plane $\sigma \geqq 0$, where $s = \sigma + it$. For this purpose (cf. [2], p. 860), a function $f(s)$ will be called uniformly almost periodic, in the half-plane $\sigma \geqq 0$, if $f(s)$ is regular and, in the usual sense (cf. [1], p. 141), is uniformly almost periodic in the open half-plane $\sigma > 0$, is continuous on the closed half-plane $\sigma \geqq 0$, and $f(it)$ $(= f(s)$ on the boundary line $\sigma = 0)$ is uniformly almost periodic for $-\infty < t < \infty$. If the variable it is replaced by s in each of the series (2), it follows from the theorem of [1], pp. 150-151, that each of the new series

$$\sum_k a_{kj} e^{-\lambda_k s} \tag{2'}$$

is the Fourier expansion of a corresponding function

$$F_0(s), F_1(s), \cdots, F_{n-1}(s), \tag{1'}$$

uniformly almost periodic for $\sigma \geqq 0$, and such that $F_h(it) = f_h(t)$, $h = 0, 1, \cdots, n-1$. Define $F_n(s)$ by

$$F_n(s) \equiv 1. \tag{5'}$$

Let (6'), (6' bis), (7') denote the relations which result by replacing t by s in (6), (6 bis), (7), respectively. Correspondingly, replace (4) by

$$\sum_{h=0}^{n} F_h(s)\, d^h x/ds^h = 0 \tag{4'}$$

and (8) by

$$\sum_{h=0}^{n} F_h(s)\, d^h x/ds^h = G(s), \tag{8'}$$

where now $x = x(s)$, the functions $F_k(s)$ are those defined above, and $G(s)$ is obtained by replacing it by s in the function $g(t) = G(it)$. If the statement of Theorem I is altered so as to correspond to the primed relations just introduced, there results, for the complex half-plane $\sigma \geqq 0$, a theorem, to be called Theorem I′, which is similar to Theorem I for the real line $-\infty < t < \infty$. That the assumptions of Theorem I′, where uniform almost periodicity refers to the half-plane $\sigma \geqq 0$, are implied by those of Theorem I, is a consequence of the preceding paragraph. Let $s = it$ (that is, $\sigma = 0$), so that the equation (4′) [or (8′)], if multiplied by i^n, becomes

$$\sum_{k=0}^{n} i^{n-k} f_k(t)\, d^k x/dt^k = 0 \quad [\text{or} = i^n g(t)].$$

It is clear that the functions $i^{k-n} f_k(t)$ and $g(t)/i^n$ have the same structure as the respective functions $f_k(t)$ and $g(t)$. Moreover, if any solution x^k of (4) is multiplied by a non-vanishing constant, the essential properties of x^k as regard to its structure and to its being a solution of (4) are unchanged.

It is understood that all derivatives dy/ds, considered above or in the sequel, are to be interpreted on the boundary line $\sigma = 0$ in the "one-sided" sense. Thus, $(dy/ds)_{s=s_0} = \lim (f(s) - f(s_0))/(s - s_0)$, where $s_0 = it_0$ is fixed and $s = \sigma + it$, $\sigma \geqq 0$. In addition, it is to be noted that, on the half-plane $\sigma \geqq 0$, the linear independence of the solutions $x^k(s)$, considered in Theorem I′ (corresponding to (6) for Theorem I), implies their linear independence on $\sigma = 0$.

The above remarks, together with an inspection of the statement of Theorem I′ when $\sigma = 0$, make it clear that Theorem I is a corollary of Theorem I′. The proof of Theorem I′ will be given in Sections 4, 5 and 6.

3. Two lemmas will first be proved.

LEMMA 1. *On the half-plane $\sigma \geqq 0$, let $f(s)$ be a uniformly almost periodic function with a Fourier expansion*

$$f(s) \sim \sum_k a_k e^{-\mu_k s}, \qquad \text{where } \mu_m \geqq \nu > 0, \tag{9}$$

and let n denote an arbitrary positive integer. Then there exist a constant c such that $|f(s)| \leqq ce^{-\nu\sigma}$ for $\sigma \geqq 0$, and $n+1$ functions $f(s) = g_0(s), g_1(s), \cdots, g_n(s)$, each uniformly almost periodic for $\sigma \geqq 0$ and possessing a Fourier expansion of the type (9) *(with the same frequencies μ_k as for $f(s)$), such that*

$$dg_k/ds = g_{k-1} \text{ (that is, } d^k g_k/ds^k = f) \text{ for } \sigma \geqq 0 \text{ and } k = 1, 2, \cdots, n.$$

Furthermore,

$$|g_k| \leqq cv^{-k}e^{-v\sigma} \text{ for } \sigma \geqq 0, \qquad k = 0, 1, \cdots, n, \tag{10}$$

and hence, in particular,

$$|g_k| \leqq cv^{-1}e^{-v\sigma} \text{ for } \sigma \geqq 0, \qquad k = 1, 2, \cdots, n. \tag{11}$$

It is easy to see that the function $f^*(s)$ defined in [2], p. 862, formula line (16), satisfies the equation $df^*/ds = f(s)$ (loc. cit., p. 861, formula line (14)) for $\sigma \geqq 0$, rather than merely for $\sigma > 0$. This fact, and the proof and successive applications of the Lemma, loc. cit., p. 861, readily imply Lemma 1 above.

LEMMA 2. *On the half-plane $\sigma \geqq 0$, let $h_0(s), h_1(s), \cdots, h_M(s)$ denote $M+1$ uniformly almost periodic functions, each having a Fourier expansion of the type* (9). *Then, by Lemma* 1 (*cf.* (10) *for* $k = 0$), *there exists a constant c such that $|h_k(s)| \leqq ce^{-v\sigma}$ for $\sigma \geqq 0$ and $k = 0, 1, \cdots, M$. Furthermore, the differential equation*

$$d^n x/ds^n = \sum_{k=0}^{M} s^k h_k(s), \qquad \sigma \geqq 0, \tag{12}$$

possesses a solution x such that

$$d^l x/ds^l = \sum_{k=0}^{M} s^k x_{kl}, \qquad l = 0, 1, \cdots, n-1; \sigma \geqq 0, \tag{13}$$

where the functions x_{kl} are uniformly almost periodic for $\sigma \geqq 0$, with Fourier expansions satisfying (9) (*with frequencies which are contained in the set of frequencies of the functions h_k*), *and are such that*

$$|x_{kl}| \leqq C_M v^{-1} c e^{-v\sigma}, \text{ for all } k \text{ and } l = 0, 1, \cdots, n-1, \tag{14}$$

where C_M denotes a constant depending only on M.

The proof of Lemma 2 depends on a number of partial integrations and on Lemma 1. If $[y]^{(m)}$ denotes a certain m-th primitive of a function y, a partial integration shows that

$$[s^k h_k]^{(1)} = s^k g_1 - k[s^{k-1} g_1]^{(1)}, \tag{15}$$

where the present h_k and g_1 are identified with the functions f and g_1 respectively, occurring in Lemma 1. Successive partial integrations applied to (15) and the use of Lemma 1 show that

$$[s^k h_k]^{(1)} = \sum_{h=0}^{k} s^h z_{kh},$$

where the functions z_{kh} are uniformly almost periodic for $\sigma \geqq 0$ and of type (9), and satisfy

$$| z_{kh} | \leqq k! \nu^{-1} c e^{-\nu\sigma}, \qquad \sigma \geqq 0.$$

Since $k \leqq M$, it is readily seen that

$$[\sum_{k=0}^{M} s^k h_k]^{(1)} = \sum_{h=0}^{M} s^h z_{kh1}, \text{ where } | z_{kh1} | \leqq (M+1)! \nu^{-1} c e^{-\nu\sigma},$$

holds for $k = 0, 1, \cdots, M$. (The properties of the functions z_{kh1} are described below.) It follows that

$$[\sum_{n=0}^{M} s^k h_k]^{(m)} = \sum_{h=0}^{M} s^h z_{khm}, \qquad m = 1, 2, \cdots, n, \tag{16}$$

where the functions z_{khm} are uniformly almost periodic for $\sigma \geqq 0$, are of type (9), and satisfy

$$| z_{khm} | \leqq [(M+1)!]^m \nu^{-m} c e^{-\nu\sigma}, \qquad m = 1, 2, \cdots, n.$$

Relations (16) and (12) now imply (13) and (14) if x is defined by $x = [\sum_{h=0}^{M} s^k h_k]^{(n)}$ and $c_M = [(M+1)!]^n$. This completes the proof of Lemma 2.

4. *Proof of Theorem* I'. The existence of the function $x = x^1(s)$, corresponding to the $x^1(t)$ of Theorem I, satisfying the differential equation

$$\sum_{h=0}^{n} F_h(s) d^h x/ds^h = 0, \qquad \sigma \geqq 0, \tag{4'}$$

will first be shown. The differential equations, determining the successive approximations $y_1, y_2, \cdots$ to the desired solution x^1, will be set up for $\sigma \geqq 0$ as follows:

$$d^n y_{m+1}/ds^n = -\sum_{h=0}^{n-1} F_h(s) d^h y_m/ds^h, \qquad m = 0, 1, 2, \cdots, \tag{17}$$

where

$$y_0(s) \equiv 1. \tag{18}$$

For $m = 0$, (17) becomes, in virtue of (18),

$$d^n y_1/ds^n = -F_0(s). \tag{19}$$

Let c denote a constant such that

$$| F_k(s) | \leqq c e^{-\sigma}, \qquad \sigma \geqq 0,\ k = 0, 1, \cdots, n-1; \tag{20}$$

cf. Lemma 1. An application of Lemma 1 shows that (19) has a solution y_1

which, along with its first $n-1$ derivatives, is uniformly almost periodic for $\sigma \geqq 0$, with a Fourier expansion (2′) satisfying (3), and such that

$$|\, d^l y_1/ds^l \,| \leqq ce^{-\sigma}, \qquad \sigma \geqq 1, l = 0, 1, \cdots, n-1. \tag{21}$$

(The above functions y_1 and $-F_0(s)$ correspond to g_n and $f(s)$ in Lemma 1.) Suppose that a function y_m has been determined so that it and its first $n-1$ derivatives are uniformly almost periodic for $\sigma \geqq 0$ and possess Fourier expansions (2′), with exponents $\lambda_k \geqq m$, and are such that

$$|\, d^l y_m/ds^l \,| \leqq (nce^{-\sigma})^m/m!, \qquad \sigma \geqq 0, l = 0, 1, \cdots, n-1. \tag{22}$$

(These conditions are clearly satisfied for $m = 1$.) Then, by (20) and (22),

$$\Big|\, \sum_{h=0}^{n-1} F_n(s)\, d^h y_m/ds^h \,\Big| \leqq (nce^{-\sigma})^{m+1}/m!.$$

Another application of Lemma 1 shows that (17) possesses a solution y_{m+1} such that it and its first $n-1$ derivatives are uniformly almost periodic for $\sigma \geqq 0$ and satisfy

$$|\, d^l y_{m+1}/ds^l \,| \leqq (nce^{-\sigma})^{m+1}/(m+1)!, \qquad \sigma \geqq 0, l = 0, 1, \cdots, n-1.$$

Thus, (22) is satisfied if m is replaced by $m+1$. Consequently, by (18), (21) and (22) (the last being valid for $m = 0, 1, 2, \cdots$),

$$\sum_{m=1}^{\infty} |\, d^l y_m/ds^l \,| \leqq CA(\sigma), \qquad \sigma \geqq 0, l = 0, 1, \cdots, n-1, \tag{24}$$

where C denotes a constant independent of l and s, and $A(\sigma) \to 0$ holds as $\sigma \to \infty$. It is seen from (24) and (17) that the inequality (24) holds also for $l = n$.

The preceding results, together with (17) and (18), show that the function x^1 defined by

$$x^1(s) = 1 + \sum_{m=1}^{\infty} y_m(s) \tag{25}$$

is a solution of (4′). Since the series of (25) is uniformly convergent for $\sigma \geqq 0$ and since each term is uniformly almost periodic, it follows that $x^1(s)$ (together with its first n derivatives, cf. (24), (4′) and (5′)) is uniformly almost periodic for $\sigma \geqq 0$.* That x^1 is bounded and $\not\equiv 0$, with 1 as its non-

* In [2], p. 864, the matrix function $W(s)$ should be defined (in the terminology occurring there) by $W(s) = \sum_0^{\infty} V_m(s)$. Since $\sum_0^{\infty} |\, V'_m(s) \,|$ is uniformly convergent, W is the solution of the differential equation under consideration for which $W - E$ has no constant term.

vanishing constant term, follows from (25) and (24), for $l = 0$. The remaining properties of $x^1(s)$ concerning its non-vanishing exponents are clear from the above construction.

5. Consider now the inhomogeneous equation

$$\sum_{h=0}^{n} F_h(s)\, d^h x/ds^h = g(s), \qquad \sigma \geqq 0, \tag{8''}$$

where $g(s)$ is not in general uniformly almost periodic but is defined by

$$g(s) = \sum_{k=0}^{M} s^k h_k, \tag{26}$$

where the functions $h_0(s), h_1(s), \cdots, h_M(s)$ of (26) are supposed to be uniformly almost periodic of type (2′) and (3). It will be shown that (8″) has a solution $x(s)$ such that

$$d^l x/ds^l = \sum_{k=0}^{M} s^k z_{kl}, \qquad l = 0, 1, \cdots, n-1, \tag{27}$$

where the functions z_{kl} are uniformly almost periodic for $\sigma \geqq 0$ and have Fourier expansions of type (2′) and (3).

Consider first the equation

$$d^n y_1/ds^n = g(s), \qquad \sigma \geqq 0. \tag{28}$$

According to Lemma 2, (28) possesses a solution y_1 such that

$$d^l y_1/ds^l = \sum_{h=0}^{M} s^h y_{1hl}, \qquad l = 0, 1, \cdots, n-1, \tag{29}$$

where the functions y_{1hl} are uniformly almost periodic for $\sigma \geqq 0$ and are of the type (2′) and (3). Consider the equation

$$d^n y_{m+1}/ds^n = -\sum_{l=0}^{n-1} F_l(s)\, d^l y_m/ds^l, \qquad m = 1, 2, \cdots, \tag{30}$$

where the functions $F_k(s)$ are given by (1′). If $m = 1$, (30) becomes

$$d^n y_2/ds^n = -\sum_{l=0}^{n-1} F_l(s)\, d^l y_1/ds^l, \tag{31}$$

where y_1 is defined by (29) for $l = 0$. By (29) and the properties of the functions (1′), (31) can be written as

$$d^n y_2/ds^n = \sum_{h=0}^{M} s^h g_h, \tag{32}$$

where the functions g_h are uniformly almost periodic for $\sigma \geqq 0$ and are of type (9) for $\nu = 2$. Hence, by Lemma 1, there exists a constant c satisfying

$$|g_k| \leqq ce^{-2\sigma} \text{ for } \sigma \geqq 0. \tag{33}$$

It follows from Lemma 2 that (32) possesses a solution y_2 such that

$$d^l y_2/ds^l = \sum_{h=0}^{M} s^h y_{2hl}, \qquad l = 0, 1, \cdots, n-1,$$

where the functions y_{2hl} are uniformly almost periodic for $\sigma \geqq 0$ and satisfy

$$|y_{2hl}| \leqq C_M ce^{-2\sigma}/2, \qquad \sigma \geqq 0, l = 0, 1, \cdots, n-1,$$

(cf. (14)). Suppose that y_m, where $m \geqq 2$, has been determined such that

$$d^l y_m/ds^l = \sum_{h=0}^{M} s^h y_{mhl}, \qquad l = 0, 1, \cdots, n-1, \tag{34}$$

where the y_{mhl} are uniformly almost periodic for $\sigma \geqq 0$ with Fourier expansions of type (9) for $\nu = m$ and are such that

$$|y_{mhl}| \leqq (C_M nc)^{m-1} e^{-m\sigma}/m!, \qquad \sigma \geqq 1, l = 0, 1, \cdots, n-1. \tag{35}$$

Then

$$-\sum_{l=0}^{n-1} F_l(s) d^l y_m/ds^l = \sum_{h=0}^{M} s^h [-\sum_{l=0}^{n-1} F_l(s) y_{mhl}], \tag{36}$$

where, by the properties of the functions (1′) and the functions y_{mhl}, the functions $[\cdots]$ of (36), where $h = 0, \cdots, M$, are uniformly almost periodic for $\sigma \geqq 0$, are of type (9) for $\nu = m + 1$, and satisfy

$$|\sum_{l=0}^{n-1} F_l(s) y_{mhl}| \leqq (nc)^m C_M{}^{m-1} e^{-(m+1)\sigma}/m!, \qquad \sigma \geqq 0.$$

An application of Lemma 2 shows that (30) possesses a solution y_{m+1} such that (34) and (35) remain valid if m is replaced by $m + 1$. The preceding results, together with (28) and (30), show that the function $x(s)$ defined by

$$x(s) = \sum_{m=1}^{\infty} y_m(s) \tag{37}$$

satisfies (8″), where $g(s)$ is defined by (26). It is clear from the properties of the function $y_1, y_2, \cdots$ that $x(s)$ and its first $n-1$ derivatives have representations of the form (27), where the z_{kl} are uniformly almost periodic of type (2′) and (3).

6. Consider finally the equation (4′). Its solution $x = x^1(s)$ (corresponding to the function $x^1(t)$ of Section 1) has already been determined

(Section 4). Suppose that N functions $x^1(s), \cdots, x^N(s)$, where $N \geqq 1$, have been determined such that $x^k(s)$ is a solution of (4′) satisfying

$$x^1(s) = x_1(s) \tag{38}$$

or, in case $2 \leqq k \leqq N$, the equations

$$x^k(s) = sx^{k-1}(s) + x_k(s) \tag{39}$$

and

$$d^l x_k/ds^l = \sum_{h=0}^{k-2} s^h w_{hl}, \qquad l = 0, 1, \cdots, n-1, \tag{40}$$

where the functions w_{hl} are uniformly almost periodic for $\sigma \geqq 0$ and of type (2′) and (3). The existence of an x^N, in the case $N = 1$, has been proved in Section 4. In line with the assertions represented by (6′), define the function $x^{N+1}(s)$, in case $n \geqq N+1$, by

$$x^{N+1}(s) = sx^N(s) + x_{N+1}(s). \tag{41}$$

(Actually, if $n = 1$, so that $n \geqq N+1$ cannot hold, the proof of Theorem I′ is already complete.) As is readily verified, in order that $x^{N+1}(s)$ be a solution of (4′), $x_{N+1}(s)$ must satisfy the equation,

$$\sum_{l=0}^{n} F_l(s)\, d^l x_{N+1}/ds^l = g(s), \tag{42}$$

where

$$g(s) = -\sum_{l=1}^{n} lF_l(s)\, d^{l-1}x^N/ds^{l-1}.$$

Hence, properties (2′) and (3) of the functions (1′), together with the properties of the functions $x^1, \cdots, x^N$, as given by (38), (39) and (40), imply that the present function $g(s)$, defined above, may be identified with that defined by (26), where $M = N-1$. The only conceivable trouble might arise from the term $nd^{n-1}x^N/ds^{n-1}$ of the last summation. However, since $n-1 \geqq N$, it follows from (38)-(40) that this function alone is of type (26) as are also the remaining terms (hence the whole) of the summation. Thus, as was shown in Section 5, (42) possesses a solution x_{N+1}, where

$$d^l x_{N+1}/ds^l = \sum_{h=0}^{N-1} s^h z_{hl}(s), \qquad l = 0, 1, \cdots, n-1, \tag{43}$$

and the functions z_{hl} are uniformly almost periodic for $\sigma \geqq 0$ and of type (2′) and (3). It follows from (38), (41) and (43) that (38), (39) and (40) remain valid if N is replaced by $N+1$. It is clear that if the functions $x^k(s)$ are defined by (41) for $k = 2, 3, \cdots, N$ (the function x^1 being that

determined in Section 4), then one obtains n solutions $x^1, \cdots, x^n$ of (4′) possessing the structures (6′), (7′) described in Theorem I′; cf. Section 2. That these n functions are independent is clear. In fact, if s is real, it is easily verified that $x^k(s)/s^{k-1} \to 1$ as $s \to \infty$. In particular, $x^1(s)$ is the only bounded solution of (4′). The assertions regarding the exponents of the functions $x_{kh}(s)$ of (7′) are clear from the proof.

PART II.

7. Results analogous to those obtained in Part I holds in case the functions (1) are absolutely convergent Laplace integrals. In fact, the following theorems will be proved:

THEOREM II. *On the real half line $0 \leqq t < \infty$, let each of the functions* (1) *be a Laplace integral of the type*

$$f_k(t) = \int_1^\infty e^{-tu} da_k(u), \qquad 0 \leqq t < \infty, k = 0, 1, \cdots, n-1, \tag{44}$$

where

$$[a_k] = \int_1^\infty |da_k(u)| < \infty, \qquad k = 0, 1, \cdots, n-1, \tag{45}$$

and define $f_n(t)$ by (5). *Then* (4) *possesses n linearly independent solutions $x = x^1(t), \cdots, x^n(t)$ which can be chosen so as to possess the recursive structure* (6), *where $x^1(t) = x_1(t)$ is defined by*

$$x^1(t) = 1 + \int_1^\infty e^{-tu} d\beta_1(u), \tag{46}$$

the remaining functions $x_k(t)$ have the structure

$$x_k(t) = \sum_{h=0}^{k-2} t^h \int_1^\infty e^{-tu} d\beta_{kh}(u), \qquad k = 2, 3, \cdots, n, \tag{47}$$

and the functions β_1, β_{kh} satisfy the relations

$$\int_1^\infty u^n |d\beta_1(u)| < \infty, \qquad \int_1^\infty |d\beta_{kh}(u)| < \infty; \tag{48}$$

finally, the spectra of the functions β_1 and β_{kh} are contained in the closure

of the set of those t-values representable as $n_1t_1 + \cdots + n_mt_m$, *where* $m, n_1, \cdots, n_m$ *are positive integers and* $t_1, \cdots, t_m$ *are points belonging to the spectra of the functions* $a_0, a_1, \cdots, a_{n-1}$; cf. [4], p. 443.

Similar results hold for the homogeneous equation (8) if g is of the same type as the functions, f_k, $k = 0, 1, \cdots, n-1$.

The proof of Theorem II proceeds along the same lines as that of Theorem I′, the multiplication of two uniformly almost periodic functions in the latter case being now replaced by the convolution of two Laplace integrals. The rôle played by the above mentioned multiplication, in being responsible for the successive increases by one of the lower bound for the exponents involved in the successive approximations, is now taken over by the "additive rule" for spectra in the case of convolutions; cf. [3], p. 335.

8. *Proof of Theorem* II. The existence of the function $x^1(t)$ will first be shown. It is easily verified that the function

$$y_1(t) = \int_1^\infty e^{-tu} d\gamma_1(u), \tag{49}$$

where $\gamma_1(u)$ is defined by

$$(-u)^n d\gamma_1(u) = -da_0(u), \tag{50}$$

satisfies, for $0 \leqq t < \infty$, the differential equation

$$d^n y_{m+1}/dt^n = -\sum_{l=0}^{n-1} f_l(t)\, d^l y_m/dt^l, \qquad y_0(t) \equiv 1, \tag{51}$$

in which $m = 0$, so that

$$d^n y_1/dt^n = -f_0(t), \qquad 0 \leqq t < \infty. \tag{52}$$

Suppose that $y_m(t)$, where $m \geqq 1$, has been determined so that

$$y_m(t) = \int_1^\infty e^{-tu} d\gamma_m(u), \tag{53}$$

where the spectrum of $\gamma_m(t)$ does not extend below m, and where (cf. (45))

$$\delta_m = \int_1^\infty u^{n-1} \,|\, d\gamma_m(u) \,|\, < \infty. \tag{54}$$

That (53) and (54) hold for $m = 1$ follows from (49), (50), and (45). By (51) and (54),

$$\sum_{l=0}^{n-1} f_l(t)\, d^l y_m/dt^l = \sum_{l=0}^{n-1} \int_1^\infty e^{-tu} d(a_l * a_{ml})(u),$$

where

$$da_{ml}(u) = (-u)^l d\gamma_m(u), \qquad l = 0, 1, \cdots, n-1,$$

and the asterisk is the symbol of convolution. If $\gamma_{m+1}(u)$ is any function satisfying

$$(-u)^n d\gamma_{m+1}(u) = -d \sum_{l=0}^{n-1} a_l * a_{lm},$$

(so that, except for a normalization constant $\gamma_{m+1}(1)$, any function $\gamma_{m+1}(u)$ is uniquely defined at its continuity points, only $\gamma_{m+1}(u-0)$ and $\gamma_{m+1}(u+0)$ being relevant; cf. [3], p. 334), it is seen that the function $y_{m+1}(s)$ defined by (53), if m is replaced by $m+1$, satisfies (51) for $0 \leqq t < \infty$ and, since the spectrum of $\gamma_{m+1}(u)$ does not extend below $m+1$, that

$$\delta_{m+1} = \int_0^\infty u^{n-1} \mid d\gamma_{m+1}(u) \mid \leqq nC\delta_m/(m+1),$$

where $C = \max([a_0], [a_1], \cdots, [a_{n-1}])$. (Use is made here of the convolution inequalities $[a_l * a_{lm}] \leqq [a_l][a_{lm}]$; cf. [4], p. 446.) Thus

$$\delta_m = O(a^m/m!), \qquad m = 1, 2, \cdots, \tag{55}$$

where a denotes some positive constant independent of m (cf. [4], p. 446). If $\beta_1(u)$ is defined by

$$\beta_1(u) = \sum_{m=1}^{\infty} \gamma_m(u), \tag{56}$$

it is clear that the spectrum of $\beta_1(t)$ has the properties described in Section 7, and that the function $x^1(t)$ defined by (46) is a non-trivial solution of (4) (in fact, $x^1(t) \to 1$ as $t \to \infty$). That

$$\int_1^\infty u^{n-1} \mid d\beta_1(u) \mid < \infty, \tag{57}$$

follows from (55) and (56). If $t > 0$, it is seen from (4) and (46) that

$$\int_1^\infty (-u)^n e^{-tu} d\beta_1(u) = -\sum_{l=0}^{n-1} \int_1^\infty e^{-tu} d(a_l * \beta_1{}^l),$$

where $\beta_1{}^l = (-u)^l d\beta_1$, $l = 0, 1, \cdots, n-1$. Hence

$$(-u)^n d\beta_1(u) = -\sum_{l=0}^{n-1} d(a_l * \beta_1{}^l)$$

and relations (45) and (57) imply, via (57) and the relation $[a_l * \beta_1{}^l] \leqq [a_l][\beta_1{}^l]$, the first inequality of (48).

The remainder of the proof will hinge upon the following lemma, corresponding to Lemma 2 of Section 3:

LEMMA 3. *On the half-line $0 \leqq t < \infty$, let $h_0(t), h_1(t), \cdots, h_M(t)$ denote $M+1$ Laplace integrals of the type*

$$h_k(t) = \int_\nu^\infty e^{-tu} d\sigma_k(u), \qquad \nu > 0, n = 0, 1, \cdots, M,$$

where $\mu = \max([\sigma_0], [\sigma_1], \cdots, [\sigma_M]) < \infty$. Then the differential equation

$$d^n x/dt^n = \sum_{k=0}^{M} t^k h_k(t), \qquad 0 \leqq t < \infty,$$

possesses, for $0 \leqq t < \infty$, a solution such that

$$d^l x/dt^l = \sum_{k=0}^{M} t^k x_{kl}, \qquad l = 0, 1, \cdots, n-1,$$

and the functions x_{kl} are such that

$$x_{kl}(t) = \int_\nu^\infty e^{-tu} d\alpha_{kl}(u), \qquad k = 0, 1, \cdots, M \text{ and } l = 0, 1, \cdots, n-1,$$

and

$$[\alpha_{kl}] \leqq C_M \mu \nu^{-1}, \qquad k = 0, 1, \cdots, M \text{ and } l = 0, 1, \cdots, n-1$$

where C_M denotes a constant depending only on M.

The proof of Lemma 3 is similar to that of Lemma 2 and can therefore be omitted.

The remainder of the proof of Theorem II will now be sketched. By use of Lemma 3 and a method similar to that used in Section 5, one shows that the inhomogeneous differential equation

$$\sum_{h=0}^{n} f_h(t) d^h x/dt^h = \sum_{k=0}^{M} t^k h_k, \qquad 0 \leqq t < \infty,$$

where the functions $h_k(t)$ are Laplace integrals of the type (44) and (45), has a solution $x(t)$ such that

$$d^l x/dt^l = \sum_{k=0}^{M} t^k z_{kl}, \qquad l = 0, 1, \cdots, n-1,$$

and the functions z_{kl} are Laplace integrals of the type (44) and (45). The application of the above results in that portion of the proof of Theorem II, corresponding to Section 6 for Theorem I′, is clear.

That Theorem II can be generalized to the complex case by replacing the real variable t, $0 \leqq t < \infty$, by $s = \sigma + it$, where $\sigma \geqq 0$ and $-\infty < t < \infty$, is seen from the remark of [4], p. 447.

THE INSTITUTE FOR ADVANCED STUDY
AND
THE JOHNS HOPKINS UNIVERSITY.

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GENERALIZATION OF A THEOREM OF MANDELBROJT.*

By JAMES A. JENKINS.

1. Some years ago S. Mandelbrojt [6] initiated the study of quasi-analytic classes of functions integrable over a finite interval. This work was continued and extended by Levin and Lifschitz [5] who considered the relationship between the admissible integral orders of zeros of such a function and the gaps occurring in its Fourier series. Let $f(t)$ be a function summable over the interval $(-\pi, \pi)$ and have the associated Fourier series

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt).$$

Let $\psi(\alpha)$ be a continuous, non-negative, non-decreasing function defined for $0 \leqq \alpha \leqq 2\pi$. Then we say that $f(t)$ has a left-hand zero of integral order $\psi(\alpha)$ at $t = \pi$ if

$$\phi(\alpha) \equiv \int_{\pi-\alpha}^{\pi} |f(t)|\, dt \leqq \psi(\alpha) \qquad (0 \leqq \alpha \leqq 2\pi).$$

Alternatively one could consider right-hand zeros of this type or such zeros situated at other points. In terms of a function derived from $\psi(\alpha)$ and a function measuring the density of the indices of non-vanishing terms of the Fourier series, Levin and Lifschitz gave a condition implying that $f(t)$ vanishes almost everywhere on the interval $(-\pi, \pi)$. I. I. Hirschman and the author gave a simple proof of a refinement of this result [2].

The question arises whether something of a similar nature can be done for functions defined on the infinite interval $(-\infty, \infty)$ and it seems there is a rather natural connection with the general principle of Wiener that a function and its Fourier transform cannot both be too small at infinity. We may suppose that $f(t)$ and $\phi(x)$ are in $L^2(-\infty, \infty)$ and are transforms of one another in the sense of Plancherel's Theorem. G. H. Hardy [1] and G. W. Morgan [7] proved results of this type where bounds were given for the modulus of each function for $|t|$ and $|x|$ large. Ingham [4] gave a related result, assuming that f vanished outside a finite interval and that an integral related to ϕ diverged.

* Received January 13, 1951.

Recently I. I. Hirschman [3] proved a result of this type, giving a bound for the modulus of f as $t \to +\infty$ and assuming a certain rate of divergence of an integral associated with ϕ. This result is significant chiefly when f tends to zero very rapidly as $t \to \infty$ and ϕ tends to zero rather slowly. Thus it appears that his result is most closely related to Ingham's and indeed does not include the results of Hardy and Morgan in any form.

The object of the present paper is to give a generalization of the preceding results. First of all the bounds for the modulus are replaced by bounds on certain integrals. Secondly the rather special assumptions placed on ϕ by Hirschman are removed so that the result obtained includes the main part of the results of Hardy and Morgan and a more general result than Ingham's in his special case. Indeed when f vanishes say for t greater than a certain value our result reduces to one direction of the fundamental result of Paley and Wiener [8], p. 16. The principal method used, as in [2] and [3], is the principle of subordination for the logarithm of the modulus of an analytic function.

2. THEOREM 1. *Let $f(t)$ and $\phi(x)$ be in $L^2(-\infty, \infty)$ and be transforms of each other in the sense of Plancherel's Theorem, i. e.,*

$$\phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-ixt} f(t)\, dt, \qquad f(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{itx} \phi(x)\, dx,$$

where equality is to be understood in the sense of the mean square limit. Let

$$U(r) = \pi^{-1} \int_{-r}^{r} (1 + x^2)^{-1} \log |\phi(x)|\, dx \tag{1}$$

tend to $-\infty$ as $r \to \infty$. Let there exist a function $V(y)$, non-decreasing for $y > 0$ such that

$$\int_{V(y)}^{\infty} e^{ty} |f(t)|\, dt = O(e^{yV(y)}) \text{ as } y \to \infty. \tag{2}$$

Then if

$$\liminf_{r \to \infty} (V(r) + U(r)) = -\infty, \tag{3}$$

f and ϕ are both almost everywhere zero.

We observe first that the condition (2) implies that

$$F(z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-itz} f(t)\, dt \qquad (z = x + iy)$$

represents a function regular for $y > 0$. It is easily deduced that

$$\lim_{y \to 0+} \| F(x+iy) - \phi(x) \|_2 = 0, \tag{4}$$

where $\| g(x) \|_2$ denotes the L^2 norm $\left(\int_{-\infty}^{\infty} | g(x) |^2 dx \right)^{\frac{1}{2}}$ of a function $g(x)$ in $L^2(-\infty, \infty)$. Moreover, for $y > 0$

$$\begin{aligned} -\int_{-\infty}^{\infty} e^{-itz} f(t)\, dt \,| &\leqq \int_{-\infty}^{V(y)} e^{ty} | f(t) |\, dt + \int_{V(y)}^{\infty} e^{ty} | f(t) |\, dt \\ &\leqq \left(\int_{-\infty}^{V(y)} e^{2ty} dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{V(y)} | f(t) |^2\, dt \right)^{\frac{1}{2}} + O(e^{yV(y)}). \end{aligned}$$

Hence $| F(z) | \leqq A y^{-\frac{1}{2}} e^{yV(y)}$, $y < 1$ and $| F(z) | \leqq A e^{yV(y)}$, $y \geqq 1$ for an appropriate constant A.

Now let us consider a circle, centre the origin, of radius $r > 1$. Let it cut off on the line $y = \epsilon (0 < \epsilon < 1)$ a chord $S(\epsilon)$. Above the chord lies an arc $C(\epsilon)$ of the circle. Together $S(\epsilon)$ and $C(\epsilon)$ bound a domain $D(\epsilon)$ which contains the point $z = i$ and in which $F(z)$ is bounded. (The bound may depend on ϵ.) Hence we can use the principle of majorization:

$$\begin{aligned} \log | F(i) | \leqq &\int_{S(\epsilon)} K_\epsilon(x, \epsilon) \log | F(x + i\epsilon) |\, dx \\ &+ \int_{C(\epsilon)} K_\epsilon(x, y) \log | F(x + iy) |\, ds \end{aligned}$$

where $K_\epsilon(x, y)$ is the kernel giving the value of a harmonic function at the point $z = i$ in terms of its values on the boundary of $D(\epsilon)$.

In the customary notation, we write

$$\log | F(x + i\epsilon) | = \log^+ | F(x + i\epsilon) | + \log^- | F(x + i\epsilon) |.$$

As $\epsilon \to 0$, $\log^+ | F(x + i\epsilon) |$ tends in the L^2 norm to $\log^+ | \phi(x) |$, each of these clearly belonging to $L^2(-\infty, \infty)$. Thus the corresponding part of the integral over $S(\epsilon)$ tends to a finite quantity which has a uniform bound for all r. Moreover by the principle of extension of domain, replacing the circular segment $D(\epsilon)$ by a half plane bounded by $y = \epsilon$, we have

$$\begin{aligned} &\int_{S(\epsilon)} \log^- | F(x + i\epsilon) |\, K_\epsilon(x, \epsilon)\, dx \\ &\qquad \leqq \int_{S(\epsilon)} \log^- | F(x + i\epsilon) |\, \pi^{-1} (1 - \epsilon) \left(x^2 + (1 - \epsilon)^2 \right)^{-1} dx. \end{aligned}$$

Using the same argument as is used to prove Fatou's Theorem [9], p. 346, we find

$$\limsup_{\epsilon\to 0+}\int_{S(\epsilon)}\log^-|F(x+i\epsilon)|\,\pi^{-1}(1-\epsilon)(x^2+(1-\epsilon)^2)^{-1}dx$$
$$\leqq \pi^{-1}\int_{-r}^{r}(1+x^2)^{-1}\log^-|\phi(x)|\,dx.$$

On the other hand, for the integral over $C(\epsilon)$ we regard the two bounds for $|F(z)|$ according as $y<1$ *or* $y\geqq 1$ and obtain the majorant expression

$$\int_{C(\epsilon)}((A+yV(y))K_\epsilon(x,y)\,ds-\int_{C(\epsilon)-C(1)}\tfrac{1}{2}(\log y)K_\epsilon(x,y)\,ds.$$

Since $V(y)$ is increasing we can, in the first integral, replace $V(y)$ by $V(r)$ and obtain a still larger result. The new expression then tends, as $\epsilon\to 0$, to $V(r)$ plus a constant independent of r. The second integral converges to a quantity readily seen to be $o(r)$. Thus we have $\log|F(i)|\leqq V(r)+U(r)+O(1)$. Hence the condition (3) implies $F(i)=0$. If we did not have $F(z)\equiv 0$ there would be an integer m such that $F(z)$ had a zero of exactly order m at $z=i$. However the same argument applied to $F(z)(z-i)^{-m}=\Phi(z)$ would imply, since $\log|z-i|$ is uniformly bounded below on our contours as $\epsilon\to 0$, that $\Phi(i)=0$. Hence $F(z)\equiv 0$ from which the conclusion of our theorem follows.

We observe in addition that for $\phi(x)$ to vanish over an interval or to behave otherwise so that the integral in (1) diverges to $-\infty$ when taken for finite r already implies that f and ϕ are both almost everywhere zero.

3. In the paper cited, Hirschman applied his result to certain classes of quasi-analytic functions. In order to do this it was necessary to restrict consideration to those classes which contain the analytic class. We will now show that this restriction is susbtantially inessential.

First we recall a few definitions concerning quasi-analytic functions. Let $C\{M_n\}$ denote the class of functions $g(t)$, defined and infinitely differentiable for $-\infty<t<\infty$ and such that

$$|g^{(n)}(t)|\leqq Ak^nM_n \qquad (-\infty<t<\infty\,;\,n=0,1,2,\cdots)$$

with A, k constants which may depend on $g(t)$. The convex regularization $\{M_n^c\}$ of the sequence $\{M_n\}$ is given by

$$T(r)=\operatorname*{Max}_{n\geqq 0}(r^n/M_n),\qquad M_n^c=\operatorname*{Max}_{r\geqq 0}(r^n/T(r)).$$

THEOREM 2. *Let $g(t)$ belong to the class $C\{M_n\}$ and satisfy*

$$\int_{V(y)}^{\infty} e^{ty} \mid g(t) \mid dt = O(e^{yV(y)}), \qquad y \to \infty,$$

with $V(y)$ non-decreasing for $y > 0$. Let

$$W(r, h) = (2/\pi) \int^{r} (1 + x^2)^{-1} \log T(x/h)\, dx$$

satisfy for some $h > k$

$$\liminf_{r \to \infty} (V(r) - W(r, h)) = -\infty. \tag{5}$$

Finally, let $\lim_{n \to \infty} (M_n^c)^{1/n} = \infty$. Then $g(t) \equiv 0$.

Indeed, let $f(t) = g(t)(\sin t)/t$ and let $\phi(x)$ be the transform of $f(t)$ in the sense of Parseval's Theorem. Then Hirschman showed that, under the given hypotheses, for every $k' > k$, $T(\mid x \mid/k')\phi(x)$ is in L^2 $(-\infty, \infty)$. Let us denote this function by $\psi(t)$ and take k' to have the value h in (5) above. Now

$$\int_{-r}^{r} (1 + x^2)^{-1} \log \mid \phi(x) \mid dx$$
$$= \int_{-r}^{r} (1 + x^2)^{-1} \log \mid \psi(x) \mid dx - \int_{-r}^{r} (1 + x^2)^{-1} \log T(\mid x \mid/h)\, dx$$

and we easily verify that $\int_{-r}^{r} (1 + x^2)^{-1} \log \mid \psi(x) \mid dx < M$, independently of r. Thus

$$\int_{-r}^{r} (1 + x^2)^{-1} \log \mid \phi(x) \mid dx < - \int_{-r}^{r} (1 + x^2)^{-1} \log T(\mid x \mid/h)\, dx + M.$$

Clearly $f(t)$ satisfies the condition (2) in Theorem 1 and the assumption (5) of the present theorem implies condition (3) of Theorem 1. Hence both $f(t)$ and $g(t)$ are identically zero since continuous.

We observe that in addition to the generalization indicated above we have replaced the assumption on the asymptotic behavior of $g(t)$ used by Hirschman by an average assumption.

THE JOHNS HOPKINS UNIVERSITY.

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THE THEOREM OF RIEMANN-ROCH ON COMPACT ANALYTIC SURFACES.*

By KUNIHIKO KODAIRA.

The main purpose of the present paper is to prove the theorem of Riemann-Roch on compact complex analytic manifolds of complex dimension 2 with Kählerian metrics [1] by means of the theory of harmonic integrals. Suppose such a manifold $\mathfrak{M}$ as given and denote for an arbitrary divisor D on $\mathfrak{M}$ the space consisting of all meromorphic functions F on $\mathfrak{M}$ which are multiples of $-D$ by $\mathfrak{F}(D)$. The problem concerning the theorem of Riemann-Roch consists in expressing the dimension of the linear space $\mathfrak{F}(D)$ in terms of the "characters" of D such as the topological intersection-number of D with itself, the virtual genus of D, etc.[2] In this paper, we shall mainly consider the case that D is an arbitrary (reducible or irreducible) curve Γ. In Section **1**, we shall first summarize the theory of harmonic currents due to G. de Rham [3] and several results of W. V. D. Hodge [4] concerning analytic manifolds with Kählerian metrics; then we shall prove a theorem concerning the existence of analytic differentials with given singularities. In Section **2**, we shall prove a theorem due to A. Weil and, by means of that theorem, introduce the notion of the characteristic divisor class $\{\mathfrak{d}\}$ on Γ which corresponds to the characteristic series of Italian geometers. In Section 3, we first introduce the linear space $\mathfrak{f}(\Gamma, \mathfrak{d})$ consisting of meromorphic functions on Γ associated with an arbitrarily fixed characteristic divisor $\mathfrak{d} \in \{\mathfrak{d}\}$ and show that the singularities of any meromorphic function $F \in \mathfrak{F}(\Gamma)$ can be represented by a meromorphic function $f \in \mathfrak{f}(\Gamma, \mathfrak{d})$; secondly, using the existence theorem in Section **1**, we prove that, for every $f \in \mathfrak{f}(\Gamma, \mathfrak{d})$, there exists on $\mathfrak{M}$ at least one additive meromorphic function which is a multiple of $-\Gamma$ and has the singularities represented by f on Γ; thirdly, calculating the period of this additive meromorphic function, we find a necessary and sufficient condition for $f \in \mathfrak{f}(\Gamma, \mathfrak{d})$ in order that f should represent the singularities of a meromorphic function $F \in \mathfrak{F}(\Gamma)$ and deduce an expression of

* Received December 20, 1950.

[1] Kähler [13].

[2] Zariski [30], p. 66.

[3] de Rham and Kodaira [20], Part I.

[4] Hodge [12].

the difference $\dim \mathfrak{f}(\Gamma, \mathfrak{d}) - \dim \mathfrak{F}(\Gamma)$ in terms of the simple differentials of the first kind on $\mathfrak{M}$. Our method applied here is a natural generalization of that of H. Weyl.[5] The dimension of $\mathfrak{f}(\Gamma, \mathfrak{d})$ is given by the theorem of Riemann-Roch on reducible curves which will be proved in Section **4**. Combining the result of Section **3** with that of Section **4**, we shall prove, in the following Section **5**, a formula expressing $\dim \mathfrak{F}(\Gamma)$ in terms of the topological intersection number of Γ with itself, the virtual genus of Γ and two non-negative integers corresponding to the index of speciality appearing in the theorem of Riemann-Roch on algebraic curves. This formula may be called the *theorem of Riemann-Roch* on compact analytic surfaces with Kählerian metrics, since it contains the theorem of Riemann-Roch for irreducible linear systems on algebraic surfaces.[6] In the last part of this Section, we shall deduce a theorem of Castelnuovo [7] concerning the superabundance of irreducible linear systems. In the following Section **6**, we first prove a formula expressing the number of linearly independent double differentials on $\mathfrak{M}$ which are multiples of $-\Gamma$ in terms of the virtual genus of Γ, the number of connected components of Γ and the number of simple Picard integrals of the first kind which are constant on each connected components of Γ, and then we deduce from that formula a theorem of F. Enriques [8] concerning the deficiency of the series cut out on a generic curve of an arbitrary irreducible linear system by its adjoint system and a theorem of F. Severi [9] concerning the superabundance of the adjoint systems of irreducible linear systems. Applying our results to algebraic surfaces, we shall prove, in the final Section **7**, several classical results of Italian geometers, e. g. a theorem of G. Castelnuovo [10] concerning the deficiency of characteristic series. A strict algebraic proof of the theorem of Riemann-Roch for arbitrary divisors on algebraic surfaces was given recently by O. Goldman.[11] In the last part of this Section, we shall show that the theorem of Riemann-Roch for arbitrary divisors on algebraic surfaces can be readily deduced from our results in the same manner as in the proof of O. Goldman.

The author wishes to express his grateful thanks to A. Weil for suggestions and encouragement given during the period of preparation of this paper. In particular, Theorem 2. 1 and the substance of its proof, the

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[5] Weyl [27], pp. 108-125.
[6] See Zariski [30], pp. 66-69.
[7] Castelnuovo [4]; see also Zariski [30], p. 71.
[8] Enriques [8]; see also Zariski [30],pp. 64-66.
[9] Severi [23]; cf. Zariski [30], p. 144.
[10] Castelnuovo [4]; see also Zariski [30], pp. 69-70.
[11] Goldman [9].

substance of the proof of Theorem 3.2 and the notion of canonical cycles introduced in Section **5** are due to A. Weil. Again, the first part of Section **3** and the first part of Section **5** were revised according to suggestions of A. Weil. The author's best thanks are also due to O. Goldman and D. Gorenstein, who gave him the opportunity to study their manuscripts before publication at a very early stage of preparation of this paper. In addition to the method of deduction of the theorem of Riemann-Roch on algebraic surfaces in Section **7**, the author is indebted to O. Goldman in many details.

1. Currents on Kählerian manifolds. Let $\mathfrak{M}^{2n}$ be a compact analytic manifold of complex dimension n with a positive definite Kählerian metric [12]

$$ds^2 = 2\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta^*}(dz^\alpha d\bar{z}^\beta),$$

where $z = (z^1, z^2, \cdots, z^n)$ denotes the *local analytic coordinates* on $\mathfrak{M}^{2n}$. Such a manifold is called a *Kählerian manifold*. Putting

$$z^\alpha = x^{2\alpha-1} + ix^{2\alpha}, \qquad (\alpha = 1, 2, \cdots, n), \tag{1.1}$$

we introduce real coordinates $x^1, x^2, \cdots, x^{2n}$; then ds^2 can be written as

$$ds^2 = \sum_{j,k=1}^{2n} g_{jk}(dx^j dx^k), \qquad (g_{jk} = g_{kj}).$$

Thus $\mathfrak{M}^{2n}$ is a $2n$-dimensional Riemann manifold with a positive definite metric ds^2. Moreover $\mathfrak{M}^{2n}$ is *orientable*. We choose the orientation of $\mathfrak{M}^{2n}$ so that the system of local coordinates introduced in (1.1) is *positive* with respect to this orientation. By a *p-form* on $\mathfrak{M}^{2n}$ we shall mean an exterior differential form

$$\Phi = \Phi(x) = (1/p!)\sum \Phi_{jk\ldots l}(x)\,dx^j dx^k \cdots dx^l$$

of rank p with *complex* coefficients $\Phi_{jk\ldots l}$ defined in a domain $\mathfrak{D} \subseteq \mathfrak{M}^{2n}$. Φ is said to be of class C^ν, if each coefficient $\Phi_{jk\ldots l}$ is a function of class C^ν of $x^1, x^2, \cdots, x^{2n}$. We denote by $d\Phi$ the *exterior derivative* of Φ and by $*\Phi$ the *adjoint form* of Φ; again the *exterior product* of two forms Φ and Ψ will be denoted by $\Phi \cdot \Psi$. The closure of the subset $\{x \mid \Phi(x) \neq 0\}$ will be called the *carrier* of Φ and be denoted by $|\Phi|$.

Now we shall introduce the notion of currents.[13] Let $\mathfrak{D} \subseteq \mathfrak{M}^{2n}$ be an arbitrary fixed domain and $\{\Phi\}$ be the linear space consisting of all $(2n-p)$-

[12] We insert here a parenthesis to designate that the product of dz^α and $d\bar{z}^\beta$ is ordinary product.

[13] de Rham and Kodaira [20], Chap. II; Schwartz [21].

forms Φ of class C^∞ with $|\Phi| \subseteq \mathfrak{D}$. Then, a *p-current* in $\mathfrak{D}$ is, by definition, a linear functional $T[\Phi]$ defined on $\{\Phi\}$ which is *continuous* in the following sense: *For an arbitrary sequence* $\{\Phi^{(1)}, \Phi^{(2)}, \cdots, \Phi^{(m)}, \cdots\}$ *of forms* $\Phi^{(m)} \in \{\Phi\}$ *such that all* $|\Phi^{(m)}|$ *are contained in one and the same compact subset* $\mathfrak{R} \subseteq \mathfrak{D}$ *covered by a single system of local coordinates* $x^1, x^2, \cdots, x^{2n}$, *we have* $T[\Phi^{(m)}] \to 0 (m \to \infty)$ *if each partial derivative* $\partial^s \Phi^{(m)}_{jk\cdots l}/\partial x^p \partial x^q \cdots \partial x^r$ *converges uniformly to zero for* $m \to \infty$. We consider every continuous p-form Ψ defined in $\mathfrak{D}$ as a current by identifying Ψ with the linear functional

$$\Psi[\Phi] = \int_{\mathfrak{D}} \Psi \cdot \Phi; \tag{1.2}$$

similarly, we consider every differentiable $(2n-p)$-chain Γ as a p-current defined by $\Gamma[\Phi] = \int_\Gamma \Phi$. Thus *the notion of currents is a generalization of both notions of differential forms and chains.* We say that the p-current T vanishes at a point $\mathfrak{p} \in \mathfrak{D}$ and write $T = 0$ at $\mathfrak{p}$ if there exists a neighborhood $\mathfrak{U}_\mathfrak{p}$ of $\mathfrak{p}$ such that $T[\Phi] = 0$ for all Φ with $|\Phi| \subset \mathfrak{U}_\mathfrak{p}$. By the *carrier* $|T|$ of T will be meant the set consisting of all points $\mathfrak{p} \in \mathfrak{D}$ such that T does *not* vanish at $\mathfrak{p}$. We get from (1.2) the identities

$$d\Psi[\Phi] = (-1)^{p+1}\Psi[d\Phi] \quad \text{and} \quad (*\Psi)[\Phi] = (-1)^p \Psi[*\Phi].$$

In view of these identities, we define the *exterior derivative* dT and the *adjoint* $*T$ of an arbitrary p-current T by

$$dT[\Phi] = (-1)^{p+1} T[d\Phi], \qquad (*T)[\Phi] = (-1)^p T[*\Phi]. \tag{1.3}$$

Then we have $ddT = 0$ and $**T = (-1)^p T$. In case T is a $(2n-p)$-chain Γ, dT coincides with the boundary $\mathfrak{B}\Gamma$ of Γ up to the sign $(-1)^{p+1}$; $d\Gamma = (-1)^{p+1}\mathfrak{B}\Gamma$. We introduce further the operators δ and Δ defined respectively as $\delta = -*d*$, $\Delta = d\delta + \delta d$. Again, the *exterior product* $T \cdot \Psi$ of a p-current T and a q-form Ψ of class C^∞ is defined as $(T \cdot \Psi)[\Phi] = T[\Psi \cdot \Phi]$. For an arbitrary p-current T and p-form Φ of class C^∞ defined on the whole manifold $\mathfrak{M}^{2n}$, we define the *inner product* of T and Φ as $(T, \Phi) = (\Phi, T) = T[*\Phi]$. In case $T = \Psi$ is a p-form of class C^∞, we have

$$(\Psi, \Phi) = \int_\mathfrak{M} \Psi \cdot *\Phi = \int_\mathfrak{M} \Phi \cdot *\Psi;$$

especially $(\Phi, \bar{\Phi}) > 0$ if Φ does not vanish identically, where $\bar{\Phi}$ denotes the *conjugate* form $(1/p!)\sum \bar{\Phi}_{jk\cdots l} dx^j dx^k \cdots dx^l$. Again, we define the *conjugate* current $\bar{T}$ of T by the formula: $\bar{T}[\bar{\Phi}] = \overline{T[\Phi]}$. A p-current T is said to be regular at a point $\mathfrak{p} \in \mathfrak{D}$, if T coincides with a p-form of class C^∞ in some

neighborhood $\mathfrak{U}_\mathfrak{p}$ of $\mathfrak{p}$; otherwise T is called *singular* at $\mathfrak{p}$. By the *singular set* of T will be meant the set of all singular points of T. The singular set of T will be denoted by $|\,T\,|_s$. Obviously $|\,T\,|_s$ is a subset of the carrier $|\,T\,|$. In case T is defined in the whole manifold $\mathfrak{M}^{2n}$, the singular set $|\,T\,|_s$ of T is compact and, for an arbitrary open set $\mathfrak{U} \supset |\,T\,|_s$, we can find a p-form Ψ of class C^∞ such that $|\,T - \Psi\,| \subset \mathfrak{U}$. Now, the *inner* product (S, T) of arbitrary p-currents, S, T in the whole manifold $\mathfrak{M}^{2n}$ can be defined, under the assumption that $|\,S\,|_s \cap |\,T\,|_s$ is empty, as follows: *Choosing p-forms Φ and Ψ of class C^∞ so that $|\,T - \Psi\,| \cap |\,S - \Phi\,|$ is empty, we put*

$$(S, T) = (\Phi, \Psi) + (\Phi, T - \Psi) + (S - \Phi, \Psi).$$

It can be easily verified that the value of (S, T) does not depend on the choice of Φ and Ψ. We have $(dS, T) = (S, \delta T)$, $(\Delta S, T) = (S, \Delta T)$; also $(*S, T) = (-1)^p (S, *T)$.

By a *harmonic p-form of the first kind* on $\mathfrak{M}^{2n}$ we shall mean a p-form e of class C^∞ satisfying $de = 0$ and $\delta e = 0$ everywhere on $\mathfrak{M}^{2n}$. *The number of linearly independent harmonic p-forms* of the first kind [14] is equal to the p-Betti number b^p of $\mathfrak{M}^{2n}$. Let $e_1, e_2, \cdots, e_b$ $(b = b^p)$ be linearly independent real harmonic p-forms of the first kind such that $(e_j, e_k) = \delta_{jk}$. By the help of these forms we associate with each p-current T the harmonic p-form $HT = \sum_k (T, e_k) e_k$, which will be called the *harmonic part* of T. The operator H is *self-adjoint* in the sense that $(HS, T) = (S, HT)$ for arbitrary S, T. Again H satisfies $*H = H*$.

THEOREM 1.1 (G. de Rham).[15] *There exists one (and only one) linear operator G mapping any p-current T (defined in the whole of $\mathfrak{M}^{2n}$) into a p-current GT which has the following properties:*

$$\Delta GT = G\Delta T = T - HT, \qquad HGT = GHT = 0.$$

This operator G satisfies $Gd = dG$, $G\delta = \delta G$; again G is *self-adjoint* in the sense that $(GS, T) = (S, GT)$ for arbitrary S, T such that $|\,S\,|_s \cap |\,T\,|_s$ is empty. The operators G and H are *real* in the sense that $\overline{GT} = G\bar{T}$, $\overline{HT} = H\bar{T}$.

THEOREM 1.2 (G. de Rham).[16] *Let T be a p-current in a domain*

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[14] Hodge [12]. See also de Rham and Kodaira [20], Part I, de Rham and Bidal [19], de Rham [18], Kodaira [14], Chap. II.

[15] de Rham and Kodaira [20], pp. 63-66.

[16] de Rham and Kodaira [20], p. 61. This theorem is a generalization of H. Weyl's fundamental lemma in his method of orthogonal projection in potential theory. See Weyl [28]; cf. also Kodaira [14], Chap. III.

$\mathfrak{D} \subseteqq \mathfrak{M}^{2n}$. *If* ΔT *is regular at a point* $\mathfrak{p} \in \mathfrak{D}$, *then* T *is also regular at* $\mathfrak{p}$. *Again, if* dT *and* δT *are both regular at* $\mathfrak{p}$, *then* T *is also regular at* $\mathfrak{p}$.

By virtue of this theorem, it follows from the identity $\Delta GT = T - HT$ that, *if* T *is regular at* $\mathfrak{p}$, *then* GT *is also regular at* $\mathfrak{p}$. Inserting $T = \Delta GT + HT$ into (S, T), we get

$$(S, T) = (dS, GdT) + (\delta S, G\delta T) + (S, HT). \tag{1.4}$$

This formula enables us to generalize the notion of inner product (S, T) to the case that $|\, dS \,|_s \cap |\, dT \,|_s$ and $|\, \delta S \,|_s \cap |\, \delta T \,|_s$ are both empty. Let Γ, Z be arbitrary chains of dimensions $p, 2n - p$, respectively. Then $(*\Gamma, Z)$ is defined by (1.4) if $|\, d\Gamma \,| \cap |\, Z \,|$ and $|\, dZ \,| \cap |\, \Gamma \,|$ are both empty; moreover, as was proved by G. de Rham,[17] $(*\Gamma, Z)$ *coincides with the topological intersection number* $I(\Gamma, Z)$ of Γ and Z:

$$I(\Gamma, Z) = (*\Gamma, Z).$$

Especially if Γ *and* Z *are both cycles, then*

$$I(\Gamma, Z) = (*\Gamma, HZ) = \int_\Gamma HZ.$$

An arbitrary p-current T can be represented as a formal differential form

$$T = \sum_{r+s=p} (1/r!s!) \sum T_{\alpha_1 \cdots \alpha_r \beta^*_1 \cdots \beta^*_s} dz^{\alpha_1} \cdots dz^{\alpha_r} d\bar{z}^{\beta_1} \cdots d\bar{z}^{\beta_s},$$

where the coefficients $T_{\alpha_1 \cdots \alpha_r \beta^*_1 \cdots \beta^*_s}$ are *distributions* on the space of local coordinates $z^1, \cdots z^n$ in the sense of L. Schwartz.[18] By means of this expression we define operators C and Λ as follows:[19]

$$CT = \sum_{r+s=p} (1/r!s!) \sum (i)^{r-s} T_{\alpha_1 \cdots \alpha_r \beta^*_1 \cdots \beta^*_s} dz^{\alpha_1} \cdots dz^{\alpha_r} d\bar{z}^{\beta_1} \cdots d\bar{z}^{\beta_s},$$

$$\Lambda T = \sum_{r+s=p-2} (1/r!s!) \sum (-1)^r i g^{\alpha\beta^*} T_{\alpha\alpha_1 \cdots \alpha_r \beta^* \beta^*_1 \cdots \beta^*_s} dz^{\alpha_1} \cdots dz^{\alpha_r} d\bar{z}^{\beta_1} \cdots d\bar{z}^{\beta_s},$$

where $g^{\alpha\beta^*}$ are the quantities defined by $\sum_\gamma g^{\alpha\gamma^*} g_{\beta\gamma^*} = \delta^\alpha{}_\beta$. CT is a p-current; ΛT is a $(p-2)$-current (in case $p = 0$ or $p = 1$ we have to put $\Lambda T = 0$). *The operators* C *and* Λ *are commutative with* Δ, G *and* H.[20] Again we have

$$CCT = (-1)^p T, \tag{1.5}$$

$$CT[\Phi] = (-1)^p T[C\Phi] \tag{1.6}$$

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[17] de Rham and Kodaira [20], p. 71.

[18] Schwartz [21].

[19] These two operators were introduced by W. V. D. Hodge. See Hodge [12], p. 171. The definition of C and Λ employed here are due to A. Weil. Cf. Weil [24].

[20] Hodge [12], pp. 165-168, Weil [24], p. 111, Eckmann and Guggenheimer [7].

for an arbitrary p-current T; furthermore we have the important formula [21]

$$\Lambda d - d\Lambda = C^{-1}\delta C. \tag{1.7}$$

A p-current T of the type $T = (1/r!t!)\sum T_{a_1\cdots a_r\beta^*_1\cdots\beta^*_t}dz^{a_1}\cdots dz^{a_r}d\bar{z}^{\beta_1}\cdots d\bar{z}^{\beta_t}$ is called a *pure p-current* [22] of type t. Obviously every p-current T can be decomposed uniquely into the sum $T = \sum_{t=0}^{p} \boldsymbol{P}_t T$ of the pure p-currents $\boldsymbol{P}_t T$ of types t $(t = 0,1,\cdots,p)$. As was proved by B. Eckmann and H. Guggenheimer,[23] the "projection operators" $\boldsymbol{P}_t$ thus defined satisfy

$$\Delta \boldsymbol{P}_t = \boldsymbol{P}_t \Delta, \tag{1.8}$$

while we get readily

$$(T, \overline{\Phi}) = \sum_{t=0}^{p} (\boldsymbol{P}_t T, \overline{\boldsymbol{P}_t \Phi}), \tag{1.9}$$

where Φ is an arbitrary p-form of class C^∞. As one readily infers from (1.8) and (1.9), the complex normalized orthogonal bases $E_1, E_2, \cdots, E_b$ of harmonic p-forms of the first kind can be chosen so that each E_k is a *pure p-form*. Consequently we have

$$H\boldsymbol{P}_t = \boldsymbol{P}_t H \tag{1.10}$$

and therefore

$$G\boldsymbol{P}_t = \boldsymbol{P}_t G. \tag{1.11}$$

A p-form $\Phi = (1/p!)\sum \Phi_{a_1\cdots a_p}dz^{a_1}\cdots dz^{a_p}$ of type 0 is said to be *holomorphic* (or *meromorphic*) at $\mathfrak{p}$, if in some neighborhood of $\mathfrak{p}$ the coefficients $\Phi_{a_1\cdots a_p}$ are holomorphic (or meromorphic) functions of $z^1, \cdots z^n$; if Φ is holomorphic (or meromorphic) at every point $\mathfrak{p}$ of a subset $\mathfrak{T} \subseteq \mathfrak{M}^{2n}$, then we say that Φ is holomorphic (or meromorphic) on $\mathfrak{T}$. A meromorphic p-form Φ is said to be *regular* at $\mathfrak{p}$ if Φ is holomorphic at $\mathfrak{p}$; otherwise Φ is said to be *singular* at $\mathfrak{p}$. By a *p-ple differential* in a domain $\mathfrak{D} \subseteq \mathfrak{M}^{2n}$ we shall mean a meromorphic p-form Φ defined in $\mathfrak{D}$ satisfying $d\Phi = 0$ in $\mathfrak{D}$ *except for the singular points of* Φ. A p-ple differential is called a *holomorphic p-ple differential* if it is a holomorphic p-form. Let Φ be an arbitrary holomorphic p-ple differential in $\mathfrak{D}$. Then since this Φ satisfies $C\Phi = i^p\Phi$ and $\Lambda\Phi = 0$, we get, using (1.7),

$$\delta\Phi = i^{-p}\delta C\Phi = i^{-p}C(\Lambda d - d\Lambda)\Phi = 0, \qquad \text{in } \mathfrak{D}.$$

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[21] Weil [24], p. 111, Eckmann and Guggenheimer [7].

[22] See Hodge [12], pp. 188-192, Eckmann and Guggenheimer [7].

[23] Eckmann and Guggenheimer [7].

thus *an arbitrary holomorphic differential* Φ *satisfies* $\delta\Phi = 0$. A holomorphic p-ple differential defined in the whole of $\mathfrak{M}^{2n}$ is called a *p-ple differential of the first kind.* The above result shows that every p-ple differential of the first kind is a harmonic p-form of the first kind.

THEOREM 1.3. *Let T be a pure p-current of type 0 in a domain $\mathfrak{D} \subset\subset \mathfrak{M}^{2n}$. If T satisfies $dT = 0$ in $\mathfrak{D}$, then T is a holomorphic p-ple differential in $\mathfrak{D}$.*

Proof. Since T satisfies $CT = i^p T$ and $\Lambda T = 0$, we get

$$\delta T = i^{-p}\delta C T = i^{-p} C(\Lambda d - d\Lambda) T = 0, \text{ in } \mathfrak{D}.$$

Hence, by virtue of Theorem 1.2, T is a p-form of class C^∞; however, since T is of type 0, T is represented as $T = (1/p!) \sum \Phi_{a_1 a_2 \cdots a_p} dz^{a_1} \cdots dz^{a_p}$. Now the relation $dT = 0$ implies that $\partial \Phi_{a_1 a_2 \cdots a_p}/\partial \bar{z}^\beta = 0$ $(\beta = 1, 2, \cdots, n)$, proving that each $\Phi_{a_1 a_2 \cdots a_p}$ is a holomorphic function of $z^1, z^2, \cdots, z^n$, q. e. d.

The integral $F(z) = \int^z \Phi$ of an arbitrary holomorphic simple differential Φ defined in $\mathfrak{D}$ is a many valued holomorphic function in $\mathfrak{D}$ and Φ coincides with its differential dF: $\Phi = dF$. The integral $\int^z \Phi$ of an arbitrary simple differential Φ defined *in the whole of* $\mathfrak{M}$ is called a *simple Picard integral.* The simple Picard integral $\int^z \Phi$ is said to be *of the first kind* if Φ is a differential of the first kind. In case Φ has any singularities, the simple Picard integral $\int^z \Phi$ is called of *the second* or *the third kind* according as it is locally single valued or it is not.[24] Let $A_1, A_2, \cdots, A_q$ be a base of the space of all simple differentials of the first kind. Then $A_1, A_2, \cdots, A_q, \bar{A}_1, \bar{A}_2, \cdots, \bar{A}_q$ *are linearly independent and constitute a base of the space of all harmonic* 1*-forms of the first kind.*[25] Thus the number q of linearly independent simple differentials of the first kind is equal to a half of the 1-Betti number b^1 of $\mathfrak{M}^{2n}$: $q = \frac{1}{2} b^1$. This number q will be called the *irregularity* of the manifold $\mathfrak{M}^{2n}$.

THEOREM 1.4. *Let T be a pure $(p+1)$-current of type 1 defined in the whole of $\mathfrak{M}^{2n}$ satisfying $dT = 0$ and $HT = 0$, p being a positive integer $\leqq n$. Then the p-current $\Theta = (d\Lambda G + i\delta G)T$ is a pure p-current of type 0 and satisfies $d\Theta = iT$; moreover Θ is a holomorphic p-ple differential in*

[24] See Zariski [30], p. 119.

[25] Hodge [12], pp. 191-192; Weil [24], pp. 111-112.

$\mathfrak{M}^{2n} - |T|$. *Let Q be an arbitrary p-current such that $T = dQ$. Then the integral of Θ over an arbitrary p-cycle $Z \subset \mathfrak{M}^{2n} - |T|$ is given by*

$$\int_Z \Theta = i(*Z, Q) + (-1)^{p-1} iQ[HZ]. \tag{1.12}$$

Proof. Since T is a pure $(p+1)$-current of type 1, GT is also a pure $(p+1)$-current of type 1, as (1.11) shows. Hence we get, using (1.7),

$$\delta GT = i^{1-p}\delta C GT = i^{1-p} C(\Lambda d - d\Lambda) GT = -i^{1-p} C d\Lambda GT,$$

and therefore

$$\Theta = (1 + i^{-p} C) d\Lambda GT. \tag{1.13}$$

ΛGT is a pure $(p-1)$-current of type 0. Consequently we have

$$d\Lambda GT = \boldsymbol{P}_0 d\Lambda GT + \boldsymbol{P}_1 d\Lambda GT.$$

Combined with (1.13), this yields immediately $\Theta = 2\boldsymbol{P}_0 d\Lambda GT$, proving that Θ is a pure p-current of type 0. On the other hand, we have

$$d\Theta = iGd\delta T = iG\Delta T = iT - iHT,$$

proving that $d\Theta = iT$. This implies that $d\Theta = 0$ in $\mathfrak{M} - |T|$. Hence, by virtue of Theorem 1.3, Θ is a holomorphic differential in $\mathfrak{M} - |T|$. The integral $\int_Z \Theta = Z[\Theta]$ can be evaluated as follows: We have

$$Z[\Theta] = iZ[\delta G dQ] = (-1)^p i(Z, *\delta G dQ) = i(d*Z, GdQ),$$

while, by (1.4), $(*Z, Q) = (d*Z, GdQ) + (*Z, HQ)$. Hence we obtain

$$Z[\Theta] = i(*Z, Q) - i(*Z, HQ) = i(*Z, Q) + (-1)^{p-1} iQ[HZ],$$

proving (1.12).

2. Characteristic divisors. Let $\mathfrak{M}$ be a compact Kählerian manifold of complex dimension 2. By a *curve* Γ on $\mathfrak{M}$ we shall mean a compact analytic curve in $\mathfrak{M}$, i. e. a compact analytic subvariety of $\mathfrak{M}$ of complex dimension 1. As is well known, Γ is decomposed into the sum: $\Gamma = \sum_{\nu=1}^{\mu} \Gamma_\nu$ of a finite number of *irreducible* curves Γ_ν. Each Γ_ν is a holomorphic image of the *non-singular model* $\boldsymbol{\Gamma}_\nu$ of Γ_ν: $\Gamma_\nu = \phi(\boldsymbol{\Gamma}_\nu)$. The direct sum $\boldsymbol{\Gamma} = \sum_{\nu=1}^{\mu} \boldsymbol{\Gamma}_\nu$ will be called the *non-singular model* of Γ. We denote points on $\boldsymbol{\Gamma}$ by $p, q, \cdots$, and points on $\mathfrak{M}$ by $\mathfrak{p}, \mathfrak{q}, \cdots$. The mapping $q \to \mathfrak{q} = \phi(q)$ from $\boldsymbol{\Gamma}$ onto Γ is one-to-one unless $\mathfrak{q}$ is a singular point of Γ. For a singular point $\mathfrak{q}$, the

inverse image $\phi^{-1}(\mathfrak{q})$ consists of a finite number of points and each point $q \in \phi^{-1}(\mathfrak{q})$ correspond one-to-one to an irreducible branch Γ_q of Γ passing through $\mathfrak{q}$. Let τ_p be a local uniformization variable on Γ with the center p and $z = (z^1, z^2)$ be a system of local coordinates on $\mathfrak{M}$ with the origin $\mathfrak{p} = \phi(p)$. Then the mapping $q \to \mathfrak{q} = \phi(q)$ which maps the neighborhood U_p of p *onto* the branch Γ_p can be written as $z^1 = \phi_p{}^1(\tau_p)$, $z^2 = \phi_p{}^2(\tau_p)$, where $\phi_p{}^1$, $\phi_p{}^2$ are *holomorphic* functions of τ_p defined in U_p satisfying $\phi_p{}^1(0) = \phi_p{}^2(0) = 0$. On the other hand, Γ is represented at each point $\mathfrak{p} \in \mathfrak{M}$ by a *minimal local equation* $R_\mathfrak{p}(z^1, z^2) = 0$, where $R_\mathfrak{p}(z^1, z^2)$ is a holomorphic function of z^1, z^2 defined in a neighborhood $\mathfrak{U}_\mathfrak{p}$ of $\mathfrak{p}$. In case $\mathfrak{p} = \phi(p)$ is a simple point of Γ, we can choose z^1, z^2, τ_p so that $R_\mathfrak{p}(z^1, z^2) = z^1$ and that $z^2 = \tau_p$ on Γ_p. In what follows we write, for simplicity's sake, $R_p(z)$ for $R_{\phi(p)}(z)$. Again, in case it is not necessary to specify the center of the local uniformization variable τ_p, we write simply τ for τ_p and ϕ^α for $\phi_p{}^\alpha$ $(\alpha = 1, 2)$.

Now we choose for each $p \in \Gamma$ a sufficiently small neighborhood U_p of p so that the closure $[\phi(U_p)]$ of $\phi(U_p)$ satisfies the following two conditions: i) $[\phi(U_p)] - \phi(p)$ contains no singular point of Γ ($\phi(p)$ may be a singular point of Γ) and ii) the local equation $R_p(z^1, z^2) = 0$ of Γ is useful at every point $\mathfrak{q}$ on $[\phi(U_p)]$. Consider two points p, q on Γ such that $U_p \cap U_q$ is not empty. Then the ratio $T_{pq}(z) = R_p(z)/R_q(z)$ is a non-vanishing holomorphic function of $z = (z^1, z^2)$ defined in some neighborhood of $[\phi(U_p \cap U_q)]$. Putting $t_{pq}(\tau) = T_{pq}(\phi^1(\tau), \phi^2(\tau))$, we get, for each ordered pair $\{p, q\}$ with $U_p \cap U_q \neq 0$, a non-vanishing holomorphic function $t_{pq} = t_{pq}(\tau)$ defined in $U_p \cap U_q$. It is clear that these functions t_{pq} satisfy the following relations:

$$t_{pq}t_{qp} = 1, \qquad \text{in } U_p \cap U_q, \tag{2.1}$$

$$t_{pq}t_{qr}t_{rp} = 1, \qquad \text{in } U_p \cap U_q \cap U_r. \tag{2.2}$$

THEOREM 2.1 (A. Weil). *We can find a system* $\{h_p | p \in \Gamma\}$ *of meromorphic functions* h_p *defined respectively in* U_p *such that*

$$h_p/h_q = t_{pq}, \qquad \text{in } U_p \cap U_q. \tag{2.3}$$

Proof. It is sufficient to consider one irreducible component, say Γ_1, of Γ. Γ_1 is a compact Riemann surface; hence it can be decomposed into the sum of a finite number of 2-simplices so that Γ_1 becomes a finite simplicial complex $\mathfrak{K}$. We denote the vertices (0-simplices) of $\mathfrak{K}$ by $p_1, p_2, \cdots, p_j, \cdots$, the oriented 1-simplex with the vertices p_j, p_k (in order) by S_{jk}, and the oriented 2-simplex with the vertices p_j, p_k, p_l by S_{jkl}. The simplical decom-

position $\mathfrak{R}$ of $\mathbf{\Gamma}_1$ determines its dual cell decomposition $\mathfrak{R}^*$ of $\mathbf{\Gamma}_1$. The 2-cell of $\mathfrak{R}^*$ which is dual to p_j will be denoted by E_j. We assume that the simplicial decomposition $\mathfrak{R}$ is cosen so that i) each E_j is contained in one of U_p, say U_{p_j} and ii) each E_j is covered by the domain of the local uniformization parameter τ_j with the center p_j. Now, choosing for each E_j a simply connected neighborhood U_j such that i) $E_j \subset U_j \subset U_{p_j}$, that ii) U_j does not intersect with any simplex S_{klm} not containing the vertex p_j and that iii) each intersection $U_j \cap U_k$ is simply connected (if it is not empty), we put

$$t_{jk} = t_{p_j p_k}, \quad \text{in } U_j \cap U_k. \tag{2.4}$$

Then we have

$$t_{jk}t_{kj} = 1, \quad \text{in } U_j \cap U_k, \tag{2.5}$$

$$t_{jk}t_{kl}t_{lj} = 1, \quad \text{in } U_j \cap U_k \cap U_l. \tag{2.6}$$

In order to prove our theorem, it is sufficient to show the existence of a system $\{h_j\}$ of meromorphic functions h_j defined respectively in U_j such that $h_j/h_k = t_{jk}$, in $U_j \cap U_k$; indeed, the desired system $\{h_p\}$ can be obtained from $\{h_j\}$ by putting $h_p = t_{pp_j}h_j$, in $U_p \cap U_j$.

Now, let us forget the definition (2.4) of t_{jk} and consider an *arbitrary* system $\{t_{jk}\}$ of holomorphic functions t_{jk} defined respectively in some neighborhood of $[U_j \cap U_k]$ on $\mathbf{\Gamma}_1$ satisfying the conditions (2.5) and (2.6). We shall say, for a moment, that two such systems $\{t_{jk}\}$ and $\{t'_{jk}\}$ are *equivalent* if t_{jk}/t'_{jk} can be represented as $t_{jk}/t'_{jk} = h_j/h_k$, in $U_j \cap U_k$, by means of a system $\{h_j\}$ of meromorphic functions h_j defined respectively in U_j. Then, it is sufficient for our purpose to show that $\{t_{jk}\}$ is *equivalent* to $\{1\}$. Since $U_j \cap U_k$ is simply connected, the logarithmic function

$$\rho_{jk} = -\rho_{kj} = (1/2\pi i)\log t_{jk} + c_{jk}$$

is determined uniquely in $U_j \cap U_k$ up to an arbitrary rational integer c_{jk}. Putting

$$c_{jkl} = \rho_{jk} + \rho_{kl} + \rho_{lj}, \qquad \text{in } U_j \cap U_k \cap U_l,$$

we get, by virtue of (2.6), rational integers c_{jkl}. These integers c_{jkl} depend on the choice of c_{jk}; but the sum

$$J = \sum_{j<k<l} (\operatorname{sgn} S_{jkl})\, c_{jkl}$$

of c_{jkl} over all simplices S_{jkl} is independent of the choice of c_{jk}, where $\operatorname{sgn} S_{jkl}$ denotes the orientation of the simplex S_{jkl}. This sum J will be called the *characteristic* index of the system $\{t_{jk}\}$ and be denoted by $J\{t_{jk}\}$.

First, we consider the case that $J=0$. In this case, we can choose c_{jk} so that $c_{jkl}=0$. In fact, the relation $J=0$ implies that the 2-cocycle $\sum_{j<k<l} c_{jkl}S_{jkl}$ is the coboundary of a 1-cochain $\sum_{j<k} c'_{jk}S_{jk}$ with integral coefficients c'_{jk}, so that we have $c_{jkl}=c'_{jk}+c'_{kl}+c'_{lj}$. Hence, replacing ρ_{jk} by $\rho'_{jk}=\rho_{jk}-c'_{jk}$, we get $\rho'_{jk}+\rho'_{kl}+\rho'_{lj}=0$. Thus we may assume without loss of generality that

$$\rho_{jk}+\rho_{kl}+\rho_{lj}=0, \qquad \text{in } U_j\cap U_k\cap U_l \tag{2.7}$$

Now, assuming (2.7), we shall prove that there exists a system $\{\eta_j\}$ of *holomorphic* functions η_j defined respectively in U_j such that $\kappa_{jk}=\rho_{jk}-\eta_j+\eta_k$ is constant in $U_j\cap U_k$. To do this, we first remark that $\mathbf{\Gamma}_1$ can be considered as a Kählerian manifold. For an arbitrary function $\psi(\tau,\bar{\tau})$ of class C^∞ on $\mathbf{\Gamma}_1$, τ being the local uniformization variables on $\mathbf{\Gamma}_1$, we denote $(\partial\psi/\partial\tau)d\tau$ by $d_\tau\psi$ and $(\partial\psi/\partial\bar{\tau})d\bar{\tau}$ by $\bar{d}_\tau\psi$. Now, by virtue of (2.7), one can readily construct functions $\psi_j(\tau,\bar{\tau})$ of class C^∞ defined respectively in U_j so that [26]

$$\psi_j(\tau,\bar{\tau})-\psi_k(\tau,\bar{\tau})=\rho_{jk}(\tau), \qquad \text{in } U_j\cap U_k. \tag{2.8}$$

Since $\rho_{jk}(\tau)$ is holomorphic, we have $\bar{d}_\tau\psi_j=\bar{d}_\tau\psi_k$ in $U_j\cap U_k$. This shows that $Q=\bar{d}_\tau\psi_j$ is a well determined 1-form of class C^∞ on $\mathbf{\Gamma}_1$; moreover, since $dQ=(\partial^2\psi_j/\partial\tau\partial\bar{\tau})d\tau d\bar{\tau}$, dQ is a pure 2-form of type 1. Putting $\Theta=(id\Lambda G-\delta G)dQ$, we get therefore a pure 1-form Θ of class C^∞ of type 0 satisfying $d\Theta=-dQ$, as Theorem 1.4 shows. It follows from $d(Q+\Theta)=0$ that there exists in each U_j a function $\alpha_j(\tau,\bar{\tau})$ of class C^∞ satisfying

$$d\alpha_j=Q+\Theta. \tag{2.9}$$

Since Q or Θ is a pure 1-form of type 1 or 0, respectively, this yields $\bar{d}_\tau\alpha_j=Q$, proving that $\bar{d}_\tau(\psi_j-\alpha_j)=0$. Hence $\eta_j=\psi_j-\alpha_j$ is a holomorphic function in U_j. On the other hand, (2.9) shows that $d(\alpha_j-\alpha_k)=0$. Hence $\kappa_{jk}=\alpha_j-\alpha_k$ is a constant. Inserting $\psi_j=\eta_j+\alpha_j$ in (2.8), we get therefore

$$\rho_{jk}(\tau)=\eta_j(\tau)-\eta_k(\tau)+\kappa_{jk}.$$

Putting $e_j=\exp 2\pi i\eta_j$, $t^0{}_{jk}=\exp 2\pi i\kappa_{jk}$, we have therefore $t_{jk}=e_j/e_k\cdot t^0{}_{jk}$. Thus the system $\{t_{jk}\}$ is equivalent to the system $\{t^0{}_{jk}\}$ consisting of the *constants* $t^0{}_{jk}$.

Since $\kappa_{jk}+\kappa_{kl}+\kappa_{lj}=0$, the sum $\sum_{j<k}\kappa_{jk}S_{jk}$ is a 1-cocycle on $\mathfrak{R}$. This

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[26] Using the fact that each ρ_{jk} is extended analytically on a neighborhood of the closure of $U_j\cap U_k$, we can construct successively $\psi_1,\psi_2,\cdots,\psi_k,\cdots$ so that $\psi_k=\psi_j+\rho_{kj}$ in $U_k\cap U_j$ for $1\leqq j\leqq k-1$. Cf. Weil [24], p. 113.

cocycle determines a *character* χ of the 1-Betti group of $\mathfrak{R}$ over the ring of integers defined as follows:

$$\chi(\gamma) = \exp 2\pi i\Big(\sum_{j<k} \gamma_{jk}\kappa_{jk}\Big)$$

for an arbitrary integral 1-cycle $\gamma = \sum_{j<k} \gamma_{jk}S_{jk}$. Now, let f be a multiplicative meromorphic function on $\mathfrak{r}_1$ belonging to the character $\chi(\gamma)$. Then, denoting by f_j an arbitrary branch of f in U_j and putting

$$f_j/f_k = \exp 2\pi i\lambda_{jk}, \qquad (\lambda_{kj} = -\lambda_{jk}),$$

we get $\chi(\gamma) = \exp 2\pi i\big(\sum_{j<k} \gamma_{jk}\lambda_{jk}\big)$. Hence $\sum_{j<k} \gamma_{jk}(\kappa_{jk} - \lambda_{jk}) \equiv 0 \bmod 1)$ for arbitrary 1-cycles $\gamma = \sum_{j<k} \gamma_{jk}S_{jk}$. This shows that, considered as a 1-cocycle with complex coefficients modulus 1, $\sum_{j<k} (\kappa_{jk} - \lambda_{jk})S_{jk}$ is cohomologous to zero. Consequently there exists a system $\{\kappa_j\}$ of constants κ_j such that $\kappa_{jk} \equiv \lambda_{jk} + \kappa_j - \kappa_k \pmod 1$. Putting $h_j = f_j \cdot \exp 2\pi i\kappa_j$, we get therefore $t^0_{jk} = h_j/h_k$, in $U_j \cap U_k$, proving that the system $\{t^0_{jk}\}$ is equivalent to $\{1\}$. Hence the system $\{t_{jk}\}$ is also equivalent to $\{1\}$.

Now, we consider the general case that $J \neq 0$. Let $\{t'_{jk}\}$ be the system defined as $t'_{jk}(\tau) = t_{jk}(\tau)\tau_j^{m_j}/\tau_k^{m_k}$, where τ_j is the local uniformization variable with the center p_j and m_j is arbitrary rational integers. Then we get

$$J\{t'_{jk}\} = J\{t_{jk}\} + \sum_j m_j. \tag{2.10}$$

In fact, by the transformation $t_{jk} \to t'_{jk}$, ρ_{jk} is transformed to

$$\rho_{jk}(\tau) = \rho_{jk}(\tau) + (1/2\pi i)\{m_j \log_k \tau_j - m_k \log_j \tau_k\},$$

where $\log_k \tau_j$ denotes a branch of $\log \tau_j$ in $U_j \cap U_k$. Hence we get

$$J\{t'_{jk}\} = J\{t_{jk}\} + (1/2\pi i)\sum_j m_j \sum_{k<l} \operatorname{sgn} S_{jkl} \cdot \{\log_k \tau_j - \log_l \tau_j\},$$

yielding immediately (2.10). By a suitable choice of the integers m_j, we get therefore a system $\{t'_{jk}\}$ with $J\{t'_{jk}\} = 0$ which is equivalent to $\{t_{jk}\}$. Thus our problem is reduced to the special case that $J = 0$, q. e. d.

The system $\{h_\mathfrak{p}\}$ introduced in the above Theorem 2.1 is not determined uniquely. Firstly, it depends on the choice of the system of minimal local equations $\{R_\mathfrak{p} = 0\}$. Let $\{R'_\mathfrak{p} = 0\}$ be another system of minimal local equations. Then

$$M_\mathfrak{p}(z^1, z^2) = R'_\mathfrak{p}(z^1, z^2)/R_\mathfrak{p}(z^1, z^2)$$

is a non-vanishing holomorphic function in a neighborhood $\mathfrak{U}_\mathfrak{p}$ of $\mathfrak{p}$, and the meromorphic functions h'_p defined by

$$h'_p(\tau) = M_{\phi(p)}(\phi^1(\tau), \phi^2(\tau)) \cdot h_p(\tau) \tag{2.11}$$

constitute a system $\{h'_p\}$ satisfying (2.3) with respect to $\{R'_\mathfrak{p}\}$. Thus, *the system $\{h_p\}$ can be considered as a "quantity" which is represented by the "coordinates" h_p with respect to the given "reference frame" $\{R_\mathfrak{p}\}$ and the "coordinates" h'_p corresponding to any other reference frame $\{R'_\mathfrak{p}\}$ is linked to h_p by the transformation* (2.11) *associated with the transition* $\{R_\mathfrak{p}\} \to \{R'_\mathfrak{p}\}$.[27] Secondly, multiplying each h_p by an arbitrary fixed meromorphic function f on $\mathbf{\Gamma}$, we get another system $\{h'_p | h'_p = fh_p\}$ satisfying (2.3) with respect to the same frame $\{R_\mathfrak{p}\}$. Conversely, for arbitrary two systems $\{h_p\}$ and $\{h'_p\}$ satisfying (2.3) with respect the same frame $\{R_\mathfrak{p}\}$, we have $h'_p/h_p = h'_q/h_q$, in $U_p \cap U_q$, proving that $f = h'_p/h_p$ is a well determined meromorphic function on $\mathbf{\Gamma}$. Thus, *for fixed $\{R_\mathfrak{p}\}$, the system $\{h_p\}$ satisfying* (2.3) *is unique up to the multiplication by an arbitrary meromorphic function f on $\mathbf{\Gamma}$.*

For an arbitrary meromorphic function

$$f(\tau_p) = \tau_p{}^m(a_0 + a_1\tau_p + a_2\tau_p{}^2 + \cdots), \qquad (a_0 \neq 0)$$

defined in U_p, we denote by $V_p(f)$ the exponent m appearing in the above expansion of f: $V_p(f) = m$. The functional V_p thus defined is called the *valuation with the center p.* In case f is a meromorphic function defined on the whole of $\mathbf{\Gamma}$, the divisor (f) of f can be written as

$$(f) = \sum_p V_p(f) \cdot p.$$

Corresponding to this, *we define the divisor $\mathfrak{d}$ of an arbitrary system $\{h_\mathfrak{p}\}$ satisfying* (2.3) *as*

$$\mathfrak{d} = \sum_p V_p(h_p) \cdot p.$$

As (2.11) shows, $\mathfrak{d}$ is independent of the choice of the reference frame $\{R_\mathfrak{p}\}$. *By a characteristic divisor of Γ we shall mean the divisor $\mathfrak{d}$ of the system $\{h_p\}$ introduced in Theorem* 2.1. Let $\mathfrak{d}'$ be the divisor of $\{h'_p\} = \{fh_p\}$; then we have $\mathfrak{d}' - \mathfrak{d} = (f)$. *Thus, the divisor class $\{\mathfrak{d}\}$ of $\mathfrak{d}$ on $\mathbf{\Gamma}$ is determined uniquely by Γ and does not depend on the choice of the systems $\{h_p\}$ and $\{R_\mathfrak{p}\}$. The divisor class $\{\mathfrak{d}\}$ will be called the characteristic divisor class*[28] *of Γ.*

The condition (2.3) can be written as

$$h_p(\tau)/h_q(\tau) = (R_p(z)/R_q(z))_{z=\phi(\tau)}. \tag{2.12}$$

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[27] See Weyl [29], pp. 13-23.

[28] The characteristic divisor class corresponds to the *characteristic series* of Italian geometers. See Remark 2 at the end of this Section.

Denoting $\partial R_p/\partial z^a$ by $\partial_a R_p$, we have therefore

$$(2.13) \qquad \partial_a R_p(\phi(\tau))/h_p(\tau) = \partial_a R_q(\phi(\tau))/h_q(\tau), \qquad \text{in } U_p \cap U_q.$$

THEOREM 2.2. *The degree of the characteristic divisor* $\mathfrak{d}$ *of* Γ *is equal to the topological intersection number* $I(\Gamma, \Gamma)$ *of the cycle* Γ *with itself*: $\deg \mathfrak{d} = I(\Gamma, \Gamma)$.

Proof. Let $\mathfrak{U}_\mathfrak{p}$ be a sufficiently small *spherical* neighborhood of a point $\mathfrak{p} \in \Gamma$ such that the local equation $R_\mathfrak{p}(z^1, z^2) = 0$ is useful in the whole of $\mathfrak{U}_\mathfrak{p}$; again, let $T \subset \mathfrak{U}_\mathfrak{p}$ be a 2-chain of class C^2 whose boundary $\mathfrak{B}T$ does not meet with Γ. Then, as one readily infers, the intersection number $I(T, \Gamma)$ is given by

$$(2.14) \qquad I(T, \Gamma) = (1/2\pi i) \int_{\mathfrak{B}T} d \log R_\mathfrak{p}(z^1, z^2).$$

Now, let

$$\mathfrak{d} = \sum_{k=1}^{\lambda} m_k \mathfrak{p}_k - \sum_{k=\lambda+1}^{\lambda+\mu} m_k \mathfrak{p}_k = \sum_p V_p(h_p)\, p$$

be the characteristic divisor determined by a system $\{h_p\}$ satisfying (2.3), where m_k are positive integers. We choose $\{h_p\}$ so that no $\mathfrak{p}_k = \phi(p_k)$ $(k = 1, 2, \cdots, \lambda, \cdots, \lambda + \mu)$ is a singular point of Γ. Again, we denote by $\{\mathfrak{p}_{\lambda+\mu+1}, \mathfrak{p}_{\lambda+\mu+2}, \cdots, \mathfrak{p}_{\lambda+\mu+\nu}\}$ the set of all singular points of Γ. We choose for each $\mathfrak{p}_k$ $(k = 1, 2, \cdots, \lambda + \mu + \nu)$ two sufficiently small neighborhoods $\mathfrak{U}_k$, $\mathfrak{U}'_k$ such that $\mathfrak{p}_k \in \mathfrak{U}'_k \subset \mathfrak{U}_k$ and that $\mathfrak{U}'_k$ is sufficiently small compared with $\mathfrak{U}_k$. Again, for each $\mathfrak{p}_k$ $(k = 1, 2, \cdots, \lambda)$, we choose $\epsilon_k > 0$ so that the neighborhood

$$U_k = \{\tau \mid |h_{p_k}(\tau)| < \epsilon_k\}$$

of p_k satisfies $U_k \supseteq [\phi^{-1}(\mathfrak{U}_k)]$ (remember that $h_{p_k}(\tau)$ vanishes at p_k). Now, consider the covariant vector field $v(\mathfrak{q}) = (v^1(\mathfrak{q}), v^2(\mathfrak{q}))$, $\mathfrak{q} \in \Gamma$, defined as follows:

$$(2.15) \qquad \begin{cases} v^\alpha(\mathfrak{q}) = \sum_\beta g^{\alpha\beta^*}(\mathfrak{q}) \overline{\partial_\beta R_{p_k}(\mathfrak{q})} \cdot h_{p_k}(q)/\epsilon_k{}^2, \\ \qquad\qquad \text{for } \mathfrak{q} \in \phi(U_k) \quad (k = 1, 2, \cdots, \lambda), \\ v^\alpha(\mathfrak{q}) = 0, \qquad \text{for } \mathfrak{q} = \mathfrak{p}_k \quad (k = \lambda + \mu + 1, \cdots, \lambda + \mu + \nu), \\ v^\alpha(\mathfrak{q}) = \sum_\beta g^{\alpha\beta^*}(\mathfrak{q}) \cdot \overline{\partial_\beta R_p(\mathfrak{q})}/\overline{h_p(q)}, \qquad \text{otherwise}, \end{cases}$$

where $q = \phi^{-1}(\mathfrak{q})$. $v(\mathfrak{q})$ is a *continuous* vector field, as one readily infers by means of (2.13) (remember that U_k is defined so that $h_{p_k}\bar{h}_{p_k} = \epsilon_k{}^2$ on the boundary of U_k). We choose $\delta > 0$ sufficiently small and construct a

continuously differentiable vector field $u(\mathfrak{p})$ defined on the whole of $\mathfrak{M}$ so that $N_{\mathfrak{q}}[u(\mathfrak{q})-v(\mathfrak{q})]<\delta$, for all $\mathfrak{q}\in\Gamma$, where $N_{\mathfrak{q}}[u]$ denotes the norm $\sum g_{\alpha\beta^*}(\mathfrak{q})u^\alpha\bar{u}^\beta$ of the vector u. $v(\mathfrak{q})$ does not vanish except for $\mathfrak{q}=\mathfrak{p}_k$ $(k=1,2,\cdots,\lambda+\mu+\nu)$. Hence, for sufficiently small $\delta>0$, $u(\mathfrak{q})$ does not vanish for $\mathfrak{q}\in\Gamma-\mathfrak{U}'_1-\mathfrak{U}'_2\cdots-\mathfrak{U}'_{\lambda+\mu+\nu}$. Now, consider the differential equation $dz^\alpha(\mathfrak{p})/dt=u^\alpha(\mathfrak{p})$, $(t\geqq 0)$, $z^\alpha(\mathfrak{p})$ being the local coordinates of the *variable* point $\mathfrak{p}\in\mathfrak{M}$, and, for each $\mathfrak{q}\in\Gamma$, denote by $\mathfrak{q}_t$ the solution of this equation satisfying the initial condition: $\mathfrak{q}_0=\mathfrak{q}$. Then, for fixed t, the mapping $\mathfrak{q}\to\mathfrak{q}_t$ is a continuous mapping and the image Γ_t of the cycle Γ by this mapping $\mathfrak{q}\to\mathfrak{q}_t$ is homotopic to Γ. Hence we have $I(\Gamma,\Gamma)=I(\Gamma,\Gamma_t)$. For sufficiently small $t>0$, Γ_t does *not* intersect with Γ outside of $\mathfrak{U}'_1\cup\mathfrak{U}'_2\cup\cdots\cup\mathfrak{U}'_{\lambda+\mu+\nu}$. Hence, in order to find the value of $I(\Gamma,\Gamma_t)$, it is sufficient to compute the intersection number $I_k(\Gamma,\Gamma_t)$ of Γ and Γ_t *in* each $\mathfrak{U}_k$, $k=1,2,\cdots,\lambda+\mu+\nu$. For $k\leqq\lambda+\mu$, Γ consists in $\mathfrak{U}_k$ of only one branch $\Gamma^{(1)}$ $(=\Gamma)$, while, for $k>\lambda+\mu$, Γ consists in $\mathfrak{U}_k$ of a finite number of branches $\Gamma^{(1)},\Gamma^{(2)},\cdots$ corresponding to points $p_k^{(1)},p_k^{(2)},\cdots,\in\phi^{-1}(\mathfrak{p}_k)$. Obviously we have $\Gamma_t=\sum_r\Gamma_t^{(r)}$ in $\mathfrak{U}_k$ and therefore

$$I_k(\Gamma,\Gamma_t)=\sum_r I_k(\Gamma,\Gamma_t^{(r)}).$$

Let $\mathfrak{S}_k$ be the (spherical) boundary of $\mathfrak{U}_k$. Then the intersection $Z_t=\mathfrak{S}_k\cdot\Gamma_t^{(r)}$ of $\mathfrak{S}_k$ with $\Gamma_t^{(r)}$ is a 1-cycle bounding the closure $[\mathfrak{U}_k\cdot\Gamma_t^{(r)}]$ of the intersection $\mathfrak{U}_k\cdot\Gamma_t^{(r)}$. Hence, using (2.14), we get

$$I_k(\Gamma,\Gamma_t^{(r)})=(1/2\pi i)\int_{Z_t}d\log R(z),$$

where $R(z)=R_{\mathfrak{p}_k}(z)$. Let $\mathfrak{q}_t\in Z_t$ be an arbitrary point on Z_t. Then we have

$$z^\alpha(\mathfrak{q}_t)=z^\alpha(\mathfrak{q})+t\cdot u^\alpha(\mathfrak{q})+o(t),\qquad(\mathfrak{q}=\mathfrak{q}_0\in\Gamma)$$

and therefore

$$R(\mathfrak{q}_t)=t\sum_\alpha\partial_\alpha R(\mathfrak{q})\cdot u^\alpha(\mathfrak{q})+o(t)=t\sum_\alpha\partial_\alpha R(\mathfrak{q})\cdot v^\alpha(\mathfrak{q})+\kappa\cdot t\delta+o(t),$$

where $R(\mathfrak{q}_t)=R(z(\mathfrak{q}_t))$ and κ is a constant not depending on $\mathfrak{q}$. Inserting (2.15) for $v^\alpha(\mathfrak{q})$, we get

$$R(\mathfrak{q}_t)=\begin{cases}t\cdot N(\mathfrak{q})h_p(\mathfrak{q})/\epsilon_k^2+\kappa t\delta+o(t), & (1\leqq k\leqq\lambda),\\ t\cdot N(\mathfrak{q})/\overline{h_p(\mathfrak{q})}+\kappa t\delta+o(t), & (k>\lambda),\end{cases}$$

where $N(\mathfrak{q})=\sum g^{\alpha\beta^*}(\mathfrak{q})\partial_\alpha R(\mathfrak{q})\overline{\partial_\beta R(\mathfrak{q})}$, $\mathfrak{q}=\phi^{-1}(\mathfrak{q})$ and $p=p_k$ or $p_k^{(r)}$ according as $k\leqq\lambda+\mu$ or $k>\lambda+\mu$. Obviously $N(\mathfrak{q})\geqq$ const. >0 uniformly for $\mathfrak{q}_t\in Z_t$. Choosing $t>0$ and $\delta>$ sufficiently small, we get therefore

$$(1/2\pi i)\int_{Z_t} d\log R(q_t) = (1/2\pi i)\int_{Z_t} d\log h_p(q) = (1/2\pi i)\int_{Z_0} d\log h_p(q)$$

$$= V_p(h_p) = \begin{cases} m_k & (1 \leqq k \leqq \lambda), \\ -m_k & (\lambda < k \leqq \lambda+\mu), \\ 0 & (\lambda+\mu < k \leqq \lambda+\mu+\nu). \end{cases}$$

Consequently we obtain

$$I(\Gamma, \Gamma_t) = \sum_k \sum_r I(\Gamma, \Gamma_t^{(r)}) = \sum_{k=1}^{\lambda} m_k - \sum_{k=\lambda+1}^{\lambda+\mu} m_k,$$

proving that $I(\Gamma, \Gamma) = \deg \mathfrak{d}$, q. e. d.

Remark 1. In terms of the theory of fibre spaces,[29] the relations between the system $\{h_p\}$, the characteristic divisor $\mathfrak{d}$ and the characteristic index J can be expressed as follows: Consider the system $\{U_q \times \mathfrak{E} \,|\, q \in \mathbf{\Gamma}\}$ of the direct products $U_q \times \mathfrak{E}$ of U_q and the *projective line* $\mathfrak{E}$ and identify $p \times \zeta_q$ in $U_q \times \mathfrak{E}$ with $p \times \zeta_r$ in $U_r \times \mathfrak{E}$ if $\zeta_q/\zeta_r = t_{qr}(p)$. Then we get from $\{U_q \times \mathfrak{E}\}$ a compact analytic manifold $\mathfrak{F}$ consisting of all points $P = p \times \zeta_q$ which is obviously a *fibre bundle* over the base space $\mathbf{\Gamma}$. We have the natural mapping $\pi: P = p \times \zeta_q \to p = \pi(P)$ of $\mathfrak{F}$ onto $\mathbf{\Gamma}$ which is called the *projection*. Obviously $\mathfrak{F}$ is determined uniquely by the system $\{R_\mathfrak{p}\}$. Now, let $\mathfrak{F}'$ be the fibre bundle determined by another system $\{R'_\mathfrak{p}\}$. Then the mapping

$$P = p \times \zeta_q \to P' = p \times \zeta'_q, \qquad \zeta'_q = \zeta_q \cdot [R'_q/R_q]_{z=\phi(p)}$$

is a *biregular mapping of* $\mathfrak{F}$ *onto* $\mathfrak{F}'$ *satisfying* $\pi(P') = \pi(P)$. Thus $\mathfrak{F}$ and $\mathfrak{F}'$ have the same *analytic structure relative to* $\mathbf{\Gamma}$. *The analytic structure of the fibre bundle* $\mathfrak{F}$ *relative to* $\mathbf{\Gamma}$ *is therefore determined uniquely by the imbedding* ϕ *of* $\mathbf{\Gamma}$ *into* $\mathfrak{M}$. A *cross-section*[30] of $\mathfrak{F}$ is, by definition, a curve C on $\mathfrak{F}$ such that π induces on C a biregular transformation of C onto $\mathbf{\Gamma}$. The subsets $C_0 = \{p \times 0 \,|\, p \in \mathbf{\Gamma}\}$ and $C_\infty = \{p \times \infty \,|\, p \in \mathbf{\Gamma}\}$ are obviously cross-sections of $\mathfrak{F}$. Now it can be easily verfied that[31] *any system* $\{h_q\}$ *satisfying* (2. 3) *corresponds one-to-one to the cross-section* $C = \{p \times h_q(p) \,|\, p \in \mathbf{\Gamma}\}$ *which differs from* C_0 *or* C_∞. By means of this cross-section C, the characteristic divisor $\mathfrak{d} = \sum_q V_q(h_q) \cdot q$ *is represented as* $\mathfrak{d} = \pi(C \cdot C_0 - C \cdot C_\infty)$, where $C \cdot C_0$, $C \cdot C_\infty$ denote the intersection products. The divisor class $\{\mathfrak{d}\}$

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[29] Cf. Weil [25], Cartan [3]. For the general theory of fibre spaces, see Chern [5].
[30] Weil [25].
[31] Weil [25].

of $\mathfrak{d}$ is an *invariant of the analytic structure* of the fibre bundle $\mathfrak{F}$ relative to $\mathbf{\Gamma}$ and, conversely, the analytic structure of $\mathfrak{F}$ relative to $\mathbf{\Gamma}$ is determined uniquely by $\{\mathfrak{d}\}$.[32] From the topological view point, the characteristic index J is the only invariant of the fibre bundle $\mathfrak{F}$. It follows from our construction of the system $\{h_p\}$ that this invariant J is represented as $J = \deg \mathfrak{d}$.

Remark 2. In case there exists on $\mathfrak{M}$ a meromorphic function F which has each component of Γ as a polar curve of multiplicity 1 (this is the case if $\mathfrak{M}$ is an algebraic surface), the system $\{h_p\}$ is obtained simply by putting $h_p(\tau) = R_p F(\phi_p(\tau))$, so that various things concerning $\{h_p\}$ become somewhat simpler. For example, the characteristic divisor can be defined as the intersection product $\mathfrak{d} = D \cdot \Gamma$ of Γ and $D = (F) + \Gamma$, where (F) denotes the divisor of F; again, since $I(\Gamma, \Gamma) = I(D, \Gamma)$, the relation $\deg \mathfrak{d} = I(\Gamma, \Gamma)$ follows immediately from $\mathfrak{d} = D \cdot \Gamma$. The relation $\mathfrak{d} = D \cdot \Gamma$ indicates that $\mathfrak{d}$ is a member of the characteristic series[33] on Γ if $\mathfrak{d}$ is a positive divisor. The system $\{h_p\}$ is in a certain sense only a substitute for such a function F.

3. An existence theorem. By a divisor on $\mathfrak{M}$ we shall mean a 2-cycle $D = \sum m_\nu \Gamma_\nu$ with integral coefficients m_ν consisting of a finite number of irreducible curves Γ_ν. Each Γ_ν associated with $m_\nu \neq 0$ is called a *component* of D. The set of all divisors on $\mathfrak{M}$ constitutes an additive group. Every meromorphic function F on $\mathfrak{M}$ which is not identically zero determines its divisor D in the well known manner (if such a function ever exists). The divisor of F will be denoted by (F). We shall say that the divisor D' is *linearly equivalent* to D and write $D' \sim D$, if there exists on $\mathfrak{M}$ a meromorphic function F such that $D' - D = (F)$. Obviously the *linear equivalence* thus defined is an equivalence relation and therefore the set of all divisors on $\mathfrak{M}$ can be decomposed into mutually disjoint equivalence classes. Each equivalence class will be called a *divisor class*; the divisor class containing D will be denoted by $\{D\}$. We shall say that $D = \sum m_\nu \Gamma_\nu$ is *positive* and write $D > 0$, if all coefficients m_ν are $\geqq 0$ and $D \neq 0$. A meromorphic function F on $\mathfrak{M}$ will be called a *multiple* of D if $(F) - D \geqq 0$ or F is identically zero. The set of all meromorphic functions which are multiples of $-D$ will be denoted by $\mathfrak{F}(D)$:

$$\mathfrak{F}(D) = \{F \mid (F) + D \geqq 0\}.$$

$\mathfrak{F}(D)$ is a linear space over the field of all complex numbers. By the *dimension* of the divisor class $\{D\}$ we shall mean the dimension of the linear

[32] Weil [25].

[33] Cf. Zariski [30], p. 26.

space $\mathfrak{F}(D)$. This definition is legitimate, since the dimension of $\mathfrak{F}(D)$ is determined uniquely by the divisor class $\{D\}$ of D. The dimension of $\{D\}$ will be denoted by $\dim\{D\}$, i. e. we put $\dim\{D\} = \dim\mathfrak{F}(D)$. Clearly an arbitrary curve $\Gamma = \sum_{\nu=1}^{\mu}\Gamma_\nu$ mentioned in Section 2 can be considered as a divisor. Our main purpose is to estimate the dimension of the divisor class $\{\Gamma\}$ of an arbitrary curve Γ.

Let Γ, $R_\mathfrak{p}$, $\phi_p{}^\alpha$, h_p, etc. have the same meaning as in *Section* 2. First we shall show that *the meromorphic function* $1/R_\mathfrak{p}$ *can be considered as a* 0*-current defined in* $\mathfrak{U}_\mathfrak{p}$, where $\mathfrak{U}_\mathfrak{p}$ is a sufficiently small neighborhood of $\mathfrak{p}$. For this purpose, we take an arbitrary 4-form Φ of class C^∞ with $|\Phi| \subset \mathfrak{U}_\mathfrak{p}$ and consider the integral $\int(1/R_\mathfrak{p})\cdot\Phi$. *In case* $\mathfrak{p}$ *is a simple point of* Γ, *the integral* $\int(1/R_\mathfrak{p})\cdot\Phi$ *converges absolutely*; then we put

$$(3.1)_0 \qquad (1/R_\mathfrak{p})[\Phi] = \int(1/R_\mathfrak{p})\cdot\Phi.$$

In case $\mathfrak{p}$ is a singular point of Γ, we define the "*principal value*" $(1/R_\mathfrak{p})[\Phi]$ of the (divergent) integral $\int(1/R_\mathfrak{p})\cdot\Phi$ as follows: Choosing the system of local coordinates (w, z) with the *origin* $\mathfrak{p}$ (we write w, z instead of z^1, z^2) so that $R_\mathfrak{p}(w, 0)$ does not vanish identically, we put

$$(3.1) \qquad (1/R_\mathfrak{p})[\Phi] = \lim_{\epsilon\to+0}\int_{|z|>\epsilon}(1/R_\mathfrak{p})\cdot\Phi.$$

The *existence of the limit* in the right hand side of (3.1) is proved as follows: $\phi^{-1}(\mathfrak{p})$ consists of a finite number of points $p_1, p_2, \cdots, p_l$ corresponding to the irreducible branches $\Gamma_1, \Gamma_2, \cdots, \Gamma_l$ of Γ passing through $\mathfrak{p}$. Corresponding to this, $R_\mathfrak{p}(w, z)$ is decomposed into a product

$$(3.2) \qquad R_\mathfrak{p}(w, z) = M(w, z)R_1(w, z)R_2(w, z)\cdots R_l(w, z), \qquad (M(0, 0) \neq 0)$$

of the "unit" M and the irreducible distinguished polynomials [34] $R_1(w, z)$, $\cdots, R_l(w, z)$ in w representing respectively the irreducible branches $\Gamma_1, \cdots, \Gamma_l$; each $R_s(w, z)$ is written therefore as

$$R_s(w, z) = w^{m_s} + B_{s1}(z)w^{m_s-1} + \cdots + B_{sm_s}(z),$$

where $B_{s\nu}(z)$ are holomorphic functions of z satisfying $B_{s\nu}(0) = 0$. For our present purpose we may assume that $M(w, z) = 1$, since the integral in the right hand side of (3.1) can be rewritten as

$$\int_{|z|>\epsilon}(1/\prod_s R_s)\cdot\Phi' \qquad \Phi' = M^{-1}\Phi.$$

[34] See Behnke and Thullen [1], p. 57, Bochner and Martin [2], Chap. IX.

Under these assumptions, we can find such a parametric representation of each Γ_s that

$$\begin{cases} w = \mathfrak{P}_s(\tau_s) = \tau_s^{k_s}(a_{s0} + a_{s1}\tau_s + \cdots), & (a_{s0} \neq 0, k_s \geqq 1), \\ z = \tau_s^{m_s}, \end{cases}$$

so that $R_\mathfrak{p}(w, z)$ is decomposed as

$$(3.3) \qquad R_\mathfrak{p}(w, z) = \prod_s \prod_{k=0}^{m_s-1} [w - \mathfrak{P}_s(\zeta_s^k \tau_s)], \qquad (\zeta_s = \exp(2\pi i/m_s)).$$

This yields immediately

$$(3.4) \qquad 1/R_\mathfrak{p}(w, z) = \sum_s \sum_{k=0}^{m_s-1} \rho_s(\zeta_s^k \tau_s)[w - \mathfrak{P}_s(\zeta_s^k \tau_s)]^{-1},$$

where

$$\rho_s(\tau_s) = 1/\partial_1 R_\mathfrak{p}(\mathfrak{P}_s(\tau_s), \tau_s^{m_s}), \qquad (\partial_1 R_\mathfrak{p} = \partial R_\mathfrak{p}/\partial w).$$

Now, putting

$$(3.5) \qquad \psi(v, z) = \int \Phi_{11^*22^*}(v + w, z) w^{-1} dw d\bar{w}$$

and $\epsilon_s = \epsilon^{1/m_s}$, we get, using (3.4),

$$\begin{aligned} &\int_{|z|>\epsilon} (1/R_\mathfrak{p}) \cdot \Phi \\ &\quad = \sum_s \sum_k \int_{|\tau_s|>\epsilon_s} \rho_s(\zeta_s^k \tau_s) \psi(\mathfrak{P}_s(\zeta_s^k \tau_s), \tau_s^{m_s}) |\tau_s|^{2m_s-2} m_s d\tau_s d\bar{\tau}_s \\ &\quad = \sum_s \int_{|\tau_s|>\epsilon_s} \rho_s(\tau_s) \psi(\mathfrak{P}_s(\tau_s), \tau_s^{m_s}) |\tau_s|^{2m_s-2} m_s^2 d\tau_s d\bar{\tau}_s, \end{aligned}$$

since

$$\int \Phi_{11^*22^*}(w, \tau_s^{m_s})[w - \mathfrak{P}_s(\tau_s)]^{-1} dw d\bar{w} = \int \Phi_{11^*22^*}(w + \mathfrak{P}_s(\tau_s), \tau_s^{m_s}) w^{-1} dw d\bar{w}.$$

As the integral in (3.5) converges absolutely, $\psi(v, z)$ is a function of class C^∞ of $v, \bar{v}, z, \bar{z}$. Consequently $\psi(\mathfrak{P}_s(\tau_s), \tau_s^{m_s})$ is a function of class C^∞ of τ_s, $\bar{\tau}_s$ and therefore *the limit*

$$\lim_{\epsilon \to +0} \int_{|\tau_s|>\epsilon} \rho_s(\tau_s) \psi(\mathfrak{P}_s(\tau_s), \tau_s^{m_s}) |\tau_s|^{2m_s-2} d\tau_s d\bar{\tau}_s$$

does exist; indeed this limit represents the *Cauchy principal value* of the (possibly divergent) integral

$$\int \rho_s \psi(\mathfrak{P}_s, \tau_s^{m_s}) |\tau_s|^{2m_s-2} d\tau_s d\bar{\tau}_s.$$

Thus we infer the existence of the limit in the right-hand side of (3.1); moreover it follows from the above result that *the limit is given by*

$$(3.6)\qquad (1/R_{\mathfrak{p}})[\Phi] = \sum_{s} \lim_{\epsilon \to +0} \int_{|\tau_s| > \epsilon} \rho_s \psi(\mathfrak{P}_s, \tau_s^{m_s}) \,|\, \tau_s \,|^{2m_s-2} m_s{}^2 d\tau_s d\bar{\tau}_s.$$

By using (3.5) and (3.6), it can be easily verified that the linear functional $(1/R_{\mathfrak{p}})[\Phi]$ defined by (3.1) satisfies the continuity condition for currents. Indeed, it follows from (3.5) that

$$| \partial^n \psi / \partial v^h \partial \bar{v}^i \partial z^k \partial \bar{z}^j | \leqq c \max | \partial^n \Phi_{11\cdots 22\cdots} / \partial w^h \partial \bar{w}^i \partial z^k \partial \bar{z}^j |,$$

c being a constant which is independent of Φ. Consequently, if $\{\Phi^{(m)}\}$ is a sequence of 4-forms of class C^∞ with $|\Phi^{(m)}| \subset \mathfrak{U}_{\mathfrak{p}}$ such that each partial derivative $\partial^n \Phi^{(m)}{}_{11\cdots 22\cdots} / \partial w^h \partial \bar{w}^i \partial z^k \partial \bar{z}^j$ converges uniformly to zero for $m \to \infty$, we get, by (3.6), $\lim_{m\to\infty} (1/R_{\mathfrak{p}})[\Phi^{(m)}] = 0$. Hence, identifying $1/R_{\mathfrak{p}}$ with the linear functional $(1/R_{\mathfrak{p}})[\Phi]$, we can consider $1/R_{\mathfrak{p}}$ as a 0-current defined in $\mathfrak{U}_{\mathfrak{p}}$. As will be proved below, *the* 0*-current* $1/R_{\mathfrak{p}}$ *thus defined is independent of the choice of the system of local coordinates* (w, z) *appearing in the definition* (3.1).

Putting $\mathfrak{Q}_{\mathfrak{p}} = (1/4\pi i)(1 + iC) d(1/R_{\mathfrak{p}})$, we introduce the 1*-current* $\mathfrak{Q}_{\mathfrak{p}}$ in $\mathfrak{U}_{\mathfrak{p}}$. Since $d(1/R_{\mathfrak{p}})$ is holomorphic in $\mathfrak{U}_{\mathfrak{p}} - \Gamma$, we have $|\mathfrak{Q}_{\mathfrak{p}}| \subset \Gamma$. Again $\mathfrak{Q}_{\mathfrak{p}}$ satisfies $C\mathfrak{Q}_{\mathfrak{p}} = -i\mathfrak{Q}_{\mathfrak{p}}$, showing that $\mathfrak{Q}_{\mathfrak{p}}$ *is a pure* 1*-current of type* 1. Let $N_{\mathfrak{p}}$ be an arbitrary holomorphic function in $\mathfrak{U}_{\mathfrak{p}}$. Then the meromorphic function $N_{\mathfrak{p}}/R_{\mathfrak{p}}$ is considered as a 0-current, since it can be interpreted as a product of the 0-current $1/R_{\mathfrak{p}}$ and the 0-form $N_{\mathfrak{p}}$. We have

$$d(N_{\mathfrak{p}}/R_{\mathfrak{p}}) = (1/R_{\mathfrak{p}}) \cdot dN_{\mathfrak{p}} + N_{\mathfrak{p}} \cdot d(1/R_{\mathfrak{p}}),$$

while the holomorphic 1-form $dN_{\mathfrak{p}}$ satisfies $(1 + iC) dN_{\mathfrak{p}} = 0$. Hence we get

$$(3.7)\qquad (1 + iC) d(N_{\mathfrak{p}}/R_{\mathfrak{p}}) = 4\pi i \mathfrak{Q}_{\mathfrak{p}} \cdot N_{\mathfrak{p}}.$$

Similarly, for an arbitrary holomorphic double differential $B_{\mathfrak{p}} = B_{\mathfrak{p}12} dz^1 dz^2$ in $\mathfrak{U}_{\mathfrak{p}}$, the double differential $B_{\mathfrak{p}}/R_{\mathfrak{p}}$ is considered as a 2-current in $\mathfrak{U}_{\mathfrak{p}}$. We have $d(B_{\mathfrak{p}}/R_{\mathfrak{p}}) = d(1/R_{\mathfrak{p}}) \cdot B_{\mathfrak{p}}$, while, since $B_{\mathfrak{p}}/R_{\mathfrak{p}}$ is a pure 2-current of type 0, $d(B_{\mathfrak{p}}/R_{\mathfrak{p}})$ is a pure 3-current of type 1. Hence we get

$$\begin{aligned} 2d(B_{\mathfrak{p}}/R_{\mathfrak{p}}) &= (1 - iC) d(B_{\mathfrak{p}}/R_{\mathfrak{p}}) = (1 - iC)\{d(1/R_{\mathfrak{p}}) \cdot B_{\mathfrak{p}}\} \\ &= \{(1 + iC) d(1/R_{\mathfrak{p}})\} \cdot B_{\mathfrak{p}} = 4\pi i \mathfrak{Q}_{\mathfrak{p}} \cdot B_{\mathfrak{p}}, \end{aligned}$$

proving that

$$(3.8)\qquad d(B_{\mathfrak{p}}/R_{\mathfrak{p}}) = 2\pi i \mathfrak{Q}_{\mathfrak{p}} \cdot B_{\mathfrak{p}}.$$

Our next task is to find an explicit expression of the 1-current $\Omega_\mathfrak{p}$. Since $R_p(\phi_p{}^1(\tau), \phi_p{}^2(\tau))$ vanishes identically, we have

$$\partial_1 R_p(\phi_p) \cdot d\phi_p{}^1 + \partial_2 R_p(\phi_p) \cdot d\phi_p{}^2 = 0,$$

where $\partial_\alpha = \partial/\partial z^\alpha$ $(\alpha = 1, 2)$ and $\phi_p = (\phi_p{}^1, \phi_p{}^2)$. Consequently, putting

$$\sigma_p = \partial\phi_p{}^2/\partial_1 R_p(\phi_p) = -\partial\phi_p{}^1/\partial_2 R_p(\phi_p),$$

we can define a *differential* σ_p defined in a neighborhood $U_p \subset \boldsymbol{\Gamma}$ of p. In case $\mathfrak{p} = \phi(p)$ is a simple point of Γ, σ_p is a holomorphic differential.

THEOREM 3.1. *For an arbitrary 3-form Ψ of class C^∞ with $|\Psi| \subset \mathfrak{U}_\mathfrak{p}$, we have*

$$\Omega_\mathfrak{p}[\Psi] = \sum_{\phi(p)=\mathfrak{p}} \lim_{\epsilon\to+0} \int_{|\tau_p|>\epsilon} \sum_\beta \Psi_{12\beta^*}(\phi_p)\sigma_p d\bar{\phi}_p{}^\beta, \tag{3.9}$$

where $\sum_{\phi(p)=\mathfrak{p}}$ denotes the sum extended over all points $p \in \phi^{-1}(\mathfrak{p})$.

The proof of this theorem will be given below. Using (3.8), we get from (3.9) immediately the following

COROLLARY. *Let $B_\mathfrak{p} = B_{12}(z)dz^1dz^2$ be a holomorphic double differential in $\mathfrak{U}_\mathfrak{p}$. Then, for an arbitrary 1-form X of class C^∞ with $|X| \subset \mathfrak{U}_\mathfrak{p}$, we have*

$$d(B_\mathfrak{p}/R_\mathfrak{p})[X] = 2\pi i \sum_{\phi(p)=\mathfrak{p}} \lim_{\epsilon\to+0} \int_{|\tau_p|>\epsilon} B_{12}(\phi_p)\sigma_p \cdot X, \tag{3.10}$$

where the symbol X in the integral sign stands for the differential

$$\sum_\beta \{X_\beta(\phi_p)d\phi_p{}^\beta + X_{\beta^*}(\phi_p)d\bar{\phi}_p{}^\beta\}$$

on Γ_p induced by X.

It is to be noted here that the expressions (3.9), (3.10) are independent of the choice of local uniformization parameters τ_p, since the limits of the integrals in the right-hand sides of (3.9), (3.10) represent the Cauchy principal values of the integrals. For an arbitrary meromorphic differential ξ, defined in U_p, we denote by $\mathrm{Res}_p[\xi]$ the residue $(1/2\pi i)\oint_p \xi$ of ξ at p, where $\oint_p$ denotes the integral extended over a sufficiently small circle $|\tau_p| = \epsilon$ with the center p. Then we have

THEOREM 3.2. *For an arbitrary holomorphic double differential $B_\mathfrak{p} = B_{12}(z)dz^1dz^2$, we have*

$$\sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[B_{12}(\phi_p)\sigma_p] = 0. \tag{3.11}$$

In order to prove the above two Theorems, we first verify the formula (3. 9) for an arbitrary *simple* point $\mathfrak{p}$ of Γ. Put, for simplicity's sake, $R = R_{\mathfrak{p}}$. Then at least one of the first partial derivatives of R, say $\partial_1 R$, does not vanish at $\mathfrak{p}$. Using (1. 3) and (1. 6), we get

$$4\pi i \mathfrak{Q}_{\mathfrak{p}}[\Psi] = (1 + iC)\, d(1/R)\,[\Psi]$$
$$= -(1/R)\,[d(1 - iC)\Psi] = -\int (1/R)\, d(1 - iC)\Psi,$$

while, since $d(1/R) \cdot (1 - iC)\Psi$ vanishes identically in $\mathfrak{U}_{\mathfrak{p}} - \Gamma$, we have

$$(1/R) \cdot d(1 - iC)\Psi = d\{(1/R)\,(1 - iC)\Psi\}, \qquad \text{in } \mathfrak{U}_{\mathfrak{p}} - \Gamma.$$

Consequently we get

$$4\pi i \mathfrak{Q}_{\mathfrak{p}}[\Psi] = -\lim_{\epsilon \to +0} \int_{|R|=\epsilon} d\{(1/R)\,(1 - iC)\Psi\} = \lim_{\epsilon \to +0} \int_{|R|=\epsilon} (1/R)\,(1 - iC)\Psi$$
$$= \lim_{\epsilon \to +0} \int_{|R|=\epsilon} 2\,(dR/R)\,\{\Psi_{122^*} - \overline{(\partial_2 R/\partial_1 R)}\,\Psi_{121^*}\}\,(1/\partial_1 R)\, dz^2 d\bar{z}^2$$
$$= 4\pi i \int \{\Psi_{122^*} - \overline{(\partial_2 R/\partial_1 R)}\,\Psi_{121^*}\}\,(1/\partial_1 R)\, dz^2 d\bar{z}^2.$$

Inserting $z^2 = \phi_p{}^2(\tau_p)$ in this expression, we obtain

$$\mathfrak{Q}_{\mathfrak{p}}[\Psi] = \int_{\Gamma} \sum_{\beta} \Psi_{12\beta^*}(\phi_p)\, \sigma_p d\bar{\phi}_p{}^{\beta};$$

thus (3. 9) is proved for the simple point $\mathfrak{p}$ of Γ, so that (3. 10) is also proved for the simple point $\mathfrak{p}$.

To prove the formula (3. 11), it is sufficient to consider the case that $\mathfrak{p}$ is a singular point of Γ. Let $\chi = \chi(z)$ be a function of class C^∞ with $|\chi| \subset \mathfrak{U}_{\mathfrak{p}}$ such that $\chi(z) = 1$ in a sufficiently small neighborhood $\mathfrak{V}_{\mathfrak{p}} \subset \mathfrak{U}_{\mathfrak{p}}$ of $\mathfrak{p}$. Then we have $|d\chi| \subset \mathfrak{U}_{\mathfrak{p}} - \mathfrak{p}$, while Γ has no singular point in $\mathfrak{U}_{\mathfrak{p}} - \mathfrak{p}$. Hence, using the formula (3. 10) *proved already for any simple point*, we get

$$d(B_{\mathfrak{p}}/R_{\mathfrak{p}})\,[d\chi] = \sum_{\phi(p)=\mathfrak{p}} 2\pi i \int_{U_p} B_{12}(\phi_p)\, \sigma_p d\chi(\phi_p).$$

Now, $B_{12}(\phi_p)\sigma_p$ is a holomorphic differential on $U_p - p$ and $\chi(\phi_p) = 1$ in the small neighborhood $U_p \cap \phi_p{}^{-1}(\mathfrak{V}_{\mathfrak{p}})$ of p. Hence we get

$$\int_{U_p} B_{12}(\phi_p)\, \sigma_p d\chi(\phi_p) = -\int_{U_p} d\{B_{12}(\phi_p)\, \sigma_p \chi(\phi_p)\}$$
$$= \oint_p B_{12}(\phi_p)\, \sigma_p = 2\pi i\, \mathrm{Res}_p[B_{12}(\phi_p)\, \sigma_p]$$

and consequently

$$2\pi i \sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[B_{12}(\phi_p)\sigma_p] = -(B_{\mathfrak{p}}/R_{\mathfrak{p}})[dd\chi] = 0,$$

proving (3.11).

Now, using (3.11), we shall prove (3.9) for an arbitrary singular point $\mathfrak{p}$ of Γ. For this purpose, we first prove the following

LEMMA 3.1. *Let $\{\xi_p | p \in \phi^{-1}(\mathfrak{p})\}$ be a system of the differentials ξ_p defined respectively on the irreducible branches Γ_p, $p \in \phi^{-1}(\mathfrak{p})$, such that*

$$\sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[B(\phi_p{}^1, \phi_p{}^2)\xi_p] = 0$$

for every holomorphic function $B(z^1, z^2)$ defined in $\mathfrak{U}_{\mathfrak{p}}$, $\mathfrak{p}$ being a singular point of Γ. Then, for an arbitrary function $\psi(z^1, z^2)$ of class C^∞ defined in $\mathfrak{U}_{\mathfrak{p}}$, we have

$$\sum_{\phi(p)=\mathfrak{p}} \lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} \psi(\phi_p{}^1, \phi_p{}^2)\xi_p = 0.$$

Proof. Let n be a positive integer such that $\tau_p{}^n \xi_p$ is holomorphic for each $p \in \phi^{-1}(\mathfrak{p})$. The function $\psi(z^1, z^2)$ can be represented in $\mathfrak{U}_{\mathfrak{p}}$ as

$$\psi(z^1, z^2) = P(z^1, z^2, \bar{z}^1, \bar{z}^2) + O(|z|^n),$$

where $P(z^1, z^2, \bar{z}^1, \bar{z}^2)$ is a polynomial in $z^1, z^2, \bar{z}^1, \bar{z}^2$ of degree $\leqq n-1$ and $|z| = (|z^1|^2 + |z^2|^2)^{\frac{1}{2}}$. Hence we have

$$\lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} \psi \cdot \xi_p = \lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} P(\phi_p{}^1, \phi_p{}^2, \bar{\phi}_p{}^1, \bar{\phi}_p{}^2) \cdot \xi_p$$

while it is clear that

$$\lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} \tau^k \bar{\tau}^m d\tau = 0, \quad \text{for } m = 1, 2, \cdots, \quad k = 0, \pm 1, \pm 2, \cdots.$$

Putting $P(z^1, z^2) = P(z^1, z^2, 0, 0)$, we get therefore

$$\begin{aligned} \sum_{\phi(p)=\mathfrak{p}} \lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} \psi \cdot \xi_p &= \sum_{\phi(p)=\mathfrak{p}} \lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} P(\phi_p{}^1, \phi_p{}^2) \cdot \xi_p \\ &= \sum_{\phi(p)=\mathfrak{p}} 2\pi i \, \mathrm{Res}_p[P(\phi_p{}^1, \phi_p{}^2) \cdot \xi_p] = 0, \end{aligned}$$

q. e. d.

Now, supposing that $R_{\mathfrak{p}}(w, z)$ is represented as (3.3), we put

$$\phi_s{}^1 = \phi_{p_s}{}^1 = \mathfrak{P}_s(\tau_s), \quad \phi_s{}^2 = \phi_{p_s}{}^2 = \tau_s{}^{m_s}, \quad \sigma_s = \sigma_{p_s} = \rho_s(\tau_s) m_s \tau_s{}^{m_s - 1} d\tau_s,$$

where $\rho_s(\tau_s) = 1/\partial_1 R_\mathfrak{p}(\phi_s)$. Then, putting $R = R_\mathfrak{p}$, we get

$$4\pi i \mathfrak{Q}_\mathfrak{p}[\Psi] = -(1/R)[d(1-iC)\Psi] = -\lim_{\epsilon\to+0}\int_{|z|>\epsilon} d\{(1/R)(1-iC)\Psi\}$$

$$= -\lim_{\epsilon\to+0}\lim_{\delta\to+0}\int_{|z|>\epsilon,\,|R|>\delta} d\{(1/R)(1-iC)\Psi\}$$

$$= -\lim_{\epsilon\to+0}\lim_{\delta\to+0}\{\int_{|z|=\epsilon,\,|R|>\delta}(1/R)(1-iC)\Psi + \int_{|z|>\epsilon,\,|R|=\delta}(1/R)(1-iC)\Psi\},$$

while, as was already proved, we have

$$\lim_{\delta\to+0}\int_{|z|>\epsilon,\,|R|=\delta}(1/R)(1-iC)\Psi = 4\pi i\sum_s\int_{|\tau_s|>\epsilon_s}\sum_\beta \Psi_{12\beta^*}(\phi_s)\sigma_s d\bar{\phi}_s{}^\beta,$$

where $\epsilon_s = \epsilon^{1/m_s}$. Hence putting $\eta(\epsilon) = \int_{|z|=\epsilon}(1/R)(1-iC)\Psi$,

$$\mathfrak{Q}_\mathfrak{p}[\Psi] = (1/4\pi i)\lim_{\epsilon\to+0}\eta(\epsilon) + \sum_s\lim_{\epsilon\to+0}\int_{|\tau_s|>\epsilon}\sum_\beta \Psi_{12\beta^*}(\phi_s)\sigma_s d\bar{\phi}_s{}^\beta.$$

Consequently, it is sufficient for our purposes to show that $\lim_{\epsilon\to+0}\eta(\epsilon) = 0$. Since $dzd\bar{z} = 0$ on the cylindrical hypersurface $|z| = \epsilon$, we have

$$\eta(\epsilon) = \int_{|z|=\epsilon}(1/R)2\Psi_{211^*}(w,z)dzdwd\bar{w}.$$

Putting $\psi(v,z) = \int 2\Psi_{211^*}(w+v,z)w^{-1}dwd\bar{w}$, we get, in the same way as in the proof of (3.6), $\eta(\epsilon) = \sum_s\sum_k\int_{|\tau_s|=\epsilon_s}\psi(\mathfrak{P}_s(\zeta_s{}^k\tau_s),\tau_s{}^{m_s})\rho_s(\zeta_s{}^k\tau_s)\tau_s{}^{m_s-1}d\tau_s$ $= \sum_s\int_{|\tau_s|=\epsilon_s}\psi(P_s(\tau_s),\tau_s{}^{m_s})\sigma_s$, showing that

$$\lim_{\epsilon\to+0}\eta(\epsilon) = \sum_s\lim_{\epsilon\to+0}\int_{|\tau_s|=\epsilon}\psi(\phi_s{}^1,\phi_s{}^2)\sigma_s.$$

$\psi(v,z)$ is obviously a function of class C^∞. Hence, by virtue of Lemma 3.1, we infer from (3.11) that $\lim_{\epsilon\to+0}\eta(\epsilon) = 0$. Thus the formula (3.9) is completely proved.

Now, we shall show that *the* 0*-current* $1/R_\mathfrak{p}$ *is determined uniquely* in the sense that it is indepedent of the choice of the local coordinates w, z appearing in the definition (3.1). This is obvious if $\mathfrak{p}$ is a simple point of Γ. In case $\mathfrak{p}$ is a singular point, $1/R_\mathfrak{p}$ is therefore determined uniquely in $\mathfrak{U}_\mathfrak{p} - \mathfrak{p}$, showing that $\mathfrak{Q}_\mathfrak{p}$ is also determined uniquely in $\mathfrak{U}_\mathfrak{p} - \mathfrak{p}$. Let $\chi = \chi(z)$ be an arbitrary function of class C^∞ with $|\chi| \subset \mathfrak{U}_\mathfrak{p} - \mathfrak{p}$, then we have, by (3.9),

$$\mathfrak{Q}_\mathfrak{p}[\chi\Psi] = \sum_{\phi(p)=\mathfrak{p}}\int_{U_p}\chi(\phi_p)\sum_\beta \Psi_{12\beta^*}(\phi_p)\sigma_p d\bar{\phi}_p{}^\beta.$$

Combined with the uniqueness of $\Omega_\mathfrak{p}$ in $\mathfrak{U}_\mathfrak{p} - \mathfrak{p}$, this shows that, in $\mathfrak{U}_\mathfrak{p} - \mathfrak{p}$, the 2-form $\sum_\beta \Psi_{12\beta^*}(\phi_p)\sigma_p d\bar{\phi}_p{}^\beta$ is independent of the choice of the local coordinates w, z, while (3.9) proves that $\Omega_\mathfrak{p}[\Psi]$ is determined uniquely by the behaviors of $\sum_\beta \Psi_{12\beta^*}(\phi_p)\sigma_p d\bar{\phi}_p{}^\beta$ in $U_p - p$, $p \in \phi^{-1}(\mathfrak{p})$. Hence we infer that the 1-current $\Omega_\mathfrak{p}$ is determined uniquely and is independent of the choice of local coordinates w, z. The 0-current $1/R_\mathfrak{p}$ is connected with $\Omega_\mathfrak{p}$ by the relation $(1 + iC)d(1/R_\mathfrak{p}) = 4\pi i\Omega_\mathfrak{p}$. In order to deduce the uniqueness of $1/R_\mathfrak{p}$ from the uniqueness of $\Omega_\mathfrak{p}$, it is sufficient therefore to show that *there exists in* $\mathfrak{U}_\mathfrak{p}$ *only one* 0*-current* T *satisfying the conditions*

$$(1 + iC)dT = 4\pi i\Omega_\mathfrak{p}, \qquad T - 1/R_\mathfrak{p} = 0, \quad \text{in } \mathfrak{U}_\mathfrak{p} - \Gamma.$$

Let T, T' be any two 0-currents satisfying the above two conditions. Then the difference $S = T - T'$ vanishes in $\mathfrak{U}_\mathfrak{p} - \Gamma$ and satisfies $CdS = idS$; thus dS is a pure 1-current of type 0, while it is obvious that $ddS = 0$. Hence, by Theorem 1.3, dS must be a holomorphic differential in $\mathfrak{U}_\mathfrak{p}$. Consequently dS vanishes in the whole of $\mathfrak{U}_\mathfrak{p}$ and therefore S must be a constant. Hence S must vanish in the whole of $\mathfrak{U}_\mathfrak{p}$, proving that $T = T'$, q. e. d.

We introduce here the notion of the *residues* of double differentials due to H. Poincaré.[35] Let $W = W_{12}(z)dz^1dz^2$ be a double differential defined in a domain $\mathfrak{D} \subseteqq \mathfrak{M}$ such that, for every point $\mathfrak{p} \in \mathfrak{D}$, $R_\mathfrak{p}W$ is holomorphic at $\mathfrak{p}$. Then, denoting by $R_\mathfrak{p}W_{12}(\phi_p(\tau_p))$ the holomorphic function of τ_p obtained from the holomorphic function $R_\mathfrak{p}W_{12}$ of z^1, z^2 by setting $z^\alpha = \phi_p{}^\alpha(\tau_p)$ $(\alpha = 1, 2)$, we infer from (3.10) that, for each $\mathfrak{p} \in \mathfrak{D}$,

$$dW[X] = 2\pi i \sum_{\phi(p)=\mathfrak{p}} \int R_pW_{12}(\phi_p(\tau_p))\sigma_pX, \tag{3.12}$$

X being an arbitrary 2-form of class C^∞ with $|X| \subset \mathfrak{U}_\mathfrak{p}$, where the value of the integral in the righ-hand side is to be interpreted as the Cauchy principal value, in case it does not converge absolutely.[36] (3.12) shows that *the differential* $R_pW_{12}(\phi_p)\sigma_p$ *defined in* U_p *is independent of the choice of* $R_\mathfrak{p}$ *and the local coordinates* z^1, z^2, U_p being the neighborhood of p introduced in Section 2. Hence we have

$$R_pW_{12}(\phi_p)\sigma_p = R_qW_{12}(\phi_q)\sigma_q, \qquad \text{in } U_p \cap U_q, \tag{3.13}$$

proving that $\xi = R_pW_{12}(\phi_p)\sigma_p$ is a well defined differential on $\mathbf{\Gamma} \cap \phi^{-1}(\mathfrak{D})$

[35] Poincaré [17].

[36] (3.12) is essentially the same as a formula due to Poincaré; see Poincaré [17], p. 350.

determined uniquely by W. This differential ξ is called the *residue of the double differential* W. We denote the residue ξ of W by $\Re(W)$, i. e. we put

$$\Re(W) = R_p W_{12}(\phi_p)\sigma_p. \tag{3.14}$$

As (3. 12) shows, *the residue* $\xi = \Re(W)$ *is characterized by the equation*

$$dW[X] = 2\pi i \int_\Gamma \xi \cdot X, \tag{3.15}$$

where X is a "variable" 2-form of class C^∞ with $|X| \subset \mathfrak{D}$. The above Theorem 3. 2 says that *the residue* ξ *of any double differential defined in* $\mathfrak{U}_\mathfrak{p}$ *satisfies* $\sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[\xi] = 0$.

The differential σ_p depends on the choice of R_p and the local coordinates z^1, z^2 in the neighborhood $\mathfrak{U}_\mathfrak{p}$ of $\mathfrak{p} = \phi(p)$. Denoting the local coordinates in $\mathfrak{U}_{\phi(p)}$ by $z_p{}^1, z_p{}^2$, we get, from (3. 13),

$$\sigma_q = [\partial(z_q{}^1, z_q{}^2)/\partial(z_p{}^1, z_p{}^2)]_{z=\phi(\tau)} t_{pq}\sigma_p, \qquad \text{in } U_p \cap U_q, \tag{3.16}$$

where $t_{pq} = [R_p(z)/R_q(z)]_{z=\phi(\tau)}$. Combined with (2. 3), this yields immediately

$$h_q\sigma_q = [\partial(z_q{}^1, z_q{}^2)/\partial(z_p{}^1, z_p{}^2)]_{z=\phi(\tau)} h_p\sigma_p, \qquad \text{in } U_p \cap U_q. \tag{3.17}$$

For an arbitrary *meromorphic* differential $\xi = f d\tau_p$ defined in a neighborhood U_p of p, we put $V_p(\xi) = V_p(f)$. By the *conductor* $\mathfrak{c}$ of Γ will be meant the divisor $\mathfrak{c}$ defined as

$$\mathfrak{c} = \sum_p c_p p, \qquad c_p = -V_p(\sigma_p).$$

It is to be noted here that $V_p(\sigma_p)$ is independent of the choice of R_p and z^1, z^2, whereas σ_p depends on them (see (3. 16)). Now we shall prove that

$$\begin{cases} c_p = 0, & \text{if } \mathfrak{p} = \phi(p) \text{ is a simple point,} \\ c_p > 0, & \text{if } \mathfrak{p} = \phi(p) \text{ is a singular point.} \end{cases}$$

To do this, we denote the system of local coordinates with the origin $\mathfrak{p}$ by (w, z). In case $\mathfrak{p} = \phi(p)$ is a simple point, we may assume that $R_\mathfrak{p}(w, z) = w$ and that $z = \tau_p$ on Γ. Hence we get $\sigma_p = d\tau_p$, proving that $c_p = 0$. In case $\mathfrak{p} = \phi(p)$ is a singular point, $\phi^{-1}(\mathfrak{p})$ consists of a finite number of points $p_1 = p, p_2, \cdots, p_l$. Let $R_\mathfrak{p}(w, z) = \prod_{s=1}^{l} R_s(w, z)$ be the decomposition of $R_\mathfrak{p}(w, z)$ into a product of the irreducible distinguished polynomials $R_s(w, z)$

in w representing respectively the irreducible branches Γ_s of Γ corresponding to p_s (see (3.2)). Then, introducing the parametric representation

$$\begin{cases} w = \mathfrak{P}_1(\tau_1) = \tau_1^{k_1}(a_0 + a_1\tau_1 + \cdots), & (a_0 \neq 0, k_1 \geqq 1), \\ z = \tau_1^{m_1} \end{cases}$$

of the branch $\Gamma_1 = \Gamma_p$, we have $d\phi_p{}^2 = m_1\tau_1^{m_1-1}d\tau_1$, while, since

$$\partial_1 R_\mathfrak{p}(\mathfrak{P}_1, \tau_1^{m_1})) = \partial_1 R_1(\mathfrak{P}_1, \tau_1^{m_1}) \cdot \prod_{s=2}^{l} R_s(\mathfrak{P}_1, \tau_1^{m_1}),$$

$V_p(\partial_1 R_\mathfrak{p}(\mathfrak{P}_1, \tau_1^{m_1}))$ is greater than $m_1 - 1$ unless $l = 1$ and $k_1 = 1$. Hence

$$c_p = V_p(\partial_1 R_\mathfrak{p}(\mathfrak{P}_1, \tau_1^{m_1})) - m_1 + 1 > 0$$

if $\mathfrak{p} = \phi(p)$ is a singular point, q. e. d. The above result shows that *the carrier* $|\mathfrak{c}|$ *of the conductor* $\mathfrak{c}$ *coincides with the inverse image* $\phi^{-1}(\mathfrak{S})$ *of the set* $\mathfrak{S}$ *consisting of all singular points of* Γ.

Now, let us consider the space $\mathfrak{F}(\Gamma)$. Let F be an arbitrary function in $\mathfrak{F}(\Gamma)$. Then R_pF is a holomorphic function in a neighborhood $\mathfrak{U}_\mathfrak{p}$ of $\mathfrak{p} = \phi(p)$ and therefore $f(\tau) = R_pF(\phi_p(\tau))/h_p(\tau)$ is a meromorphic function of τ in U_p, where $R_pF(\phi_p(\tau))$ denotes the holomorphic function of τ obtained from the holomorphic function R_pF of z^1, z^2 by putting $z^\alpha = \phi_p{}^\alpha(\tau)$ $(\alpha = 1, 2)$. Moreover, since by (2.12), $R_pF(\phi_p)/R_qF(\phi_q) = h_p/h_q$, in $U_p \cap U_q$, we have $R_pF(\phi_p)/h_p = R_qF(\phi_q)/h_q$, in $U_p \cap U_q$. This shows that $f = R_pF(\phi_p)/h_p$ *is a well defined meromorphic function on* $\boldsymbol{\Gamma}$. *We denote this function* f *by* $\mathfrak{S}_h(F)$, i. e. *we put* $\mathfrak{S}_h(F) = R_pF(\phi_p)/h_p$. The linear operator $\mathfrak{S}_h$ thus defined depends on the choice of the system $\{h_p\}$ (the subscript h in $\mathfrak{S}_h$ indicates this fact). Since $fh_p = R_pF(\phi_p)$ is holomorphic in U_p, the function $f = \mathfrak{S}_h(F)$ is a multiple of $-\mathfrak{d}$, i. e.

$$(\alpha) \qquad (f) + \mathfrak{d} \geqq 0.$$

Again, at each singular point $\mathfrak{p}$ of Γ, f satisfies

$$(\beta)^* \qquad \sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[B_{12}(\phi_p)fh_p\sigma_p] = 0$$

for an arbitrary holomorphic double differential $B_{12}dz^1dz^2$ defined in some (not fixed) neighborhood of $\mathfrak{p}$; indeed, we get, by Theorem 3.2,

$$\sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[B_{12}fh_p\sigma_p] = \sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[B_{12}R_\mathfrak{p}F\sigma_p] = 0,$$

since $R_\mathfrak{p}F$ is holomorphic. The above condition $(\beta)^*$ is obviously equivalent to the following condition

$$(\beta) \qquad \sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[(\phi_p{}^1)^k(\phi_p{}^2)^l fh_p\sigma_p] = 0, \qquad (k, l = 0, 1, 2, \cdots).$$

We denote by $\mathfrak{f}(\Gamma, \mathfrak{d})$ *the linear space consisting of all meromorphic functions* f *on* $\mathbf{\Gamma}$ *satisfying the above conditions* (α) *and* (β). Then the above results show that $F \to f = \mathfrak{S}_h(F)$ *is a linear transformation mapping* $\mathfrak{F}(\Gamma)$ *into* $\mathfrak{f}(\Gamma, \mathfrak{d})$; *moreover,* F *is determined uniquely up to an additive constant by* $f = \mathfrak{S}_h(F)$. Indeed, if $f = \mathfrak{S}_h(F) = R_p F/h_p$ vanishes identically, then the holomorphic function $R_p F$ must be divisible by R_p and therefore F is holomorphic everywhere on $\mathfrak{M}$; hence F must be a constant.

Now, suppose an arbitrary meromorphic function $f \in \mathfrak{f}(\Gamma, \mathfrak{d})$ as given. Then, does there exist a meromorphic function $F \in \mathfrak{F}(\Gamma)$ such that $\mathfrak{S}_h(F) = f$? In order to solve this question, it is convenient to introduce the notion of *additive* meromorphic functions. A many valued meromorphic function $F(z)$ on $\mathfrak{M}$ will be called *additive*, if, by the analytic continuation along an arbitrary closed continuous curve γ on $\mathfrak{M}$, $F(z)$ is changed into $F(z) + c_\gamma$, where c_γ is a constant depending on γ. An arbitrary additive meromorphic function is represented as the integral $\int^z dF$ of the *simple differential* dF; thus an additive meromorphic function is a simple Picard integral of the first or the second kind according as it is everywhere holomorphic or it is not. Let $\mathfrak{A}(\Gamma)$ be the linear space consisting of all *additive* meromorphic functions F on $\mathfrak{M}$ which are multiples of $-\Gamma$ in the sense that, for every point $\mathfrak{p}$ on $\mathfrak{M}$, $R_\mathfrak{p} F$ is holomorphic in some neighborhood of $\mathfrak{p}$. Then it can be shown in the same way as above that, for each $F \in \mathfrak{A}(\Gamma)$, $f = \mathfrak{S}_h(F) = R_p F/h_p$ is a (single valued) meromorphic function belonging to $\mathfrak{f}(\Gamma, \mathfrak{d})$. Furthermore it follows from (3.7) and (3.9) that $f = \mathfrak{S}_h(F)$ is related with F by the relation

$$(3.18) \qquad (1 + iC)\,dF[\Psi] = 4\pi i \int_{\mathbf{\Gamma}} \sum_\beta \Psi_{12\beta\bullet}(\phi_p) f h_p \sigma_p d\bar{\phi}_p{}^\beta,$$

where Ψ is an arbitrary 3-form of class C^∞ and the value of the integral in the right-hand side is to be considered as the Cauchy principal value if it does not converge absolutely. Now, we shall prove that, *for every* $f \in \mathfrak{f}(\Gamma, \mathfrak{d})$, *there exists at least one additive meromorphic function* $F \in \mathfrak{A}(\Gamma)$ *such that* $\mathfrak{S}_h(F) = f$. For this purpose, we introduce, in view of (3.18), the 1-current Q_f defined as

$$(3.19) \qquad Q_f[\Psi] = -\int_{\mathbf{\Gamma}} \sum_\beta \Psi_{12\beta\bullet}(\phi_p) f h_p \sigma_p d\bar{\phi}_p{}^\beta,$$

where Ψ is a "variable" 3-form of class C^∞ and the value of the integral in the right-hand side is to be considered as the Cauchy principal value in case

it does not converge absolutely. It is to be noted here that we have, by virtue of (3.17),

$$\sum_{\beta} \Psi_{12\beta^*}(\phi_p) f h_p \sigma_p d\bar{\phi}_p{}^{\beta} = \sum_{\beta} \Psi_{12\beta^*}(\phi_q) f h_q \sigma_q d\bar{\phi}_q{}^{\beta}, \qquad \text{in } U_p \cap U_q,$$

so that $\sum_{\beta} \Psi_{12\beta^*}(\phi_p) f h_p \sigma_p d\bar{\phi}_p{}^{\beta}$ is a well defined 2-form on $\mathbf{\Gamma}$.

It is obvious that *the carrier* $|Q_f|$ *of the* 1-*current* Q_f is Γ. Again Q_f *is a pure* 1-*current of type* 1, as (3.19) shows. Now we shall prove that dQ_f satisfies

$$CdQ_f = dQ_f. \tag{3.20}$$

To do this, let Φ be an arbitrary 2-form of class C^{∞}. Then we have

$$dQ_f[\Phi] = Q_f[d\Phi] = -\int_{\mathbf{\Gamma}} \sum_{\beta} (\partial_1 \Phi_{2\beta^*} - \partial_2 \Phi_{1\beta^*} + \partial_{\beta^*} \Phi_{12}) f h_p \sigma_p d\bar{\phi}_p{}^{\beta}.$$

Considering $\Phi_{12}(\phi_p) f h_p \sigma_p$ as a 1-form on $\mathbf{\Gamma}$, we get

$$d(\Phi_{12}(\phi_p) f h_p \sigma_p) = -\sum_{\beta} \partial_{\beta^*} \Phi_{12} f h_p \sigma_p d\bar{\phi}_p{}^{\beta}$$

and consequently

$$\int_{\mathbf{\Gamma}} \sum_{\beta} \partial_{\beta^*} \Phi_{12} f h_p \sigma_p d\bar{\phi}_p{}^{\beta} = -\int_{\mathbf{\Gamma}} d(\Phi_{12} f h_p \sigma_p) = \sum_{\mathfrak{p}} \sum_{\phi(p)=\mathfrak{p}} \lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} \Phi_{12} f h_p \sigma_p,$$

where the sum $\sum_{\mathfrak{p}}$ is to be extended over all singular points of Γ; while, as Lemma 3.1 shows, the condition $(\beta)^*$ implies that

$$\sum_{\phi(p)=\mathfrak{p}} \lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} \Phi_{12} f h_p \sigma_p = 0.$$

Hence we obtain

$$dQ_f[\Phi] = -\int_{\mathbf{\Gamma}} \sum_{\beta} (\partial_1 \Phi_{2\beta^*} - \partial_2 \Phi_{1\beta^*}) f h_p \sigma_p d\bar{\phi}_p{}^{\beta},$$

proving that $dQ_f[C\Phi] = dQ_f[\Phi]$. Combined with (1.6), this yields immediately (3.20). Thus dQ_f *is a pure* 2-*current of type* 1.

Now, we put $\theta_f = (d\Lambda G + i\delta G) dQ_f$. Then, by virtue of Theorem 1.4, θ_f is a holomorphic simple differential in $\mathfrak{M} - \Gamma$ and the integral of θ_f over an arbitrary 1-cycle $\gamma \subset \mathfrak{M} - \Gamma$ is given by

$$\int_{\gamma} \theta_f = iQ_f[H\gamma]. \tag{3.21}$$

Putting $F_f(z) = 2\pi \int^z \theta_f$, we get therefore an additive holomorphic function

F_f in $\mathfrak{M} - \Gamma$. Now we shall show that, for every $\mathfrak{p} \in \Gamma$, $R_{\mathfrak{p}}F_f$ is holomorphic in $\mathfrak{U}_{\mathfrak{p}}$. First, assume that $\mathfrak{p}$ is a simple point of Γ and put $p = \phi^{-1}(\mathfrak{p})$. Then we can choose the system of local coordinates (w, z) with the origin $\mathfrak{p}$ so that $R_{\mathfrak{p}}(w, z) = w$ and that $z = \tau_p$ on Γ. The holomorphic function $f(\tau)h_p(\tau)$ defined on $\Gamma \cap \mathfrak{U}_{\mathfrak{p}}$ can be extended to a holomorphic function $N(w, z)$ defined in $\mathfrak{U}_{\mathfrak{p}}$, e. g. we get such $N(w, z)$ by putting simply $N(w, z) = f(z)h_p(z)$. Using (3. 7) and (3. 9), we infer immediately that the 0-current $N/R_{\mathfrak{p}}$ in $\mathfrak{U}_{\mathfrak{p}}$ satisfies

$$(1 + iC)d(N/R_{\mathfrak{p}}) = -4\pi i Q_f. \tag{3.22}$$

Now, consider the difference

$$T = 2\pi\theta_f - \tfrac{1}{2}(1 - iC)d(N/R_{\mathfrak{p}}). \tag{3.23}$$

This difference T is a pure 1-current of type 0 in $\mathfrak{U}_{\mathfrak{p}}$ and, since, by Theorem 1. 4, θ_f satisfies $d\theta_f = idQ_f$, it follows from (3. 22) that T satisfies $dT = 0$. Hence, by Theorem 1. 3, T is a holomorphic differential in $\mathfrak{U}_{\mathfrak{p}}$. The integral

$$L(w, z) = \int^{(w,z)} T$$

is therefore a holomorphic function in $\mathfrak{U}_{\mathfrak{p}}$. On the other hand, we have

$$\tfrac{1}{2}(1 - iC)d(N/R_{\mathfrak{p}}) = d(N/R_{\mathfrak{p}}), \qquad \text{in } \mathfrak{U}_{\mathfrak{p}} - \Gamma.$$

Hence we get, from (3. 23),

$$F_f = N/R_{\mathfrak{p}} + L + \text{const.}, \qquad \text{in } \mathfrak{U}_{\mathfrak{p}} - \Gamma. \tag{3.24}$$

This shows that $R_{\mathfrak{p}}F_f$ is a holomorphic function in $\mathfrak{U}_{\mathfrak{p}}$. Moreover (3. 24) yields immediately $R_{\mathfrak{p}}F(0, z) = N(0, z) = f(z)h_p(z)$, proving that

$$R_{\mathfrak{p}}F(\phi_p)/h_p = f, \quad \text{in } \mathfrak{U}_{\mathfrak{p}}. \tag{3.25}$$

Now, consider the case that $\mathfrak{p}$ is a singular point of Γ. Then it follows from the above result that $R_{\mathfrak{p}}F_f$ is holomorphic in $\mathfrak{U}_{\mathfrak{p}} - \mathfrak{p}$. Hence, by a theorem of Hartogs,[37] $R_{\mathfrak{p}}F_f$ must be holomorphic in the whole of $\mathfrak{U}_{\mathfrak{p}}$. Thus F_f is a function belonging to $\mathfrak{A}(\Gamma)$. Furthermore it follows from (3. 25) that F_f satisfies $\mathfrak{S}_h(F_f) = f$. Obviously every $F \in \mathfrak{A}(\Gamma)$ satisfying $\mathfrak{S}_h(F) = f$ can be represented by means of the base $\{A_1, A_2, \cdots, A_q\}$ of simple differentials of the first kind as

$$F(z) = F_f(z) + \int^z \sum_{\nu=1}^{q} \eta_\nu A_\nu + \text{const.},$$

since $F - F_f$ is holomorphic everywhere in $\mathfrak{M}$. Thus we obtain the following

[37] Hartogs [11].

THEOREM 3.3. *The linear mapping* $F \to f = \mathfrak{S}_h(F)$ *maps* $\mathfrak{A}(\Gamma)$ *onto* $\mathfrak{f}(\Gamma, \mathfrak{d})$. *Every* $F \in \mathfrak{A}(\Gamma)$ *can be written by means of* $f = \mathfrak{S}_h(F)$ *as*

$$F(z) = \int^z (2\pi\theta_f + \sum_{\nu=1}^{q} \eta_\nu A_\nu) + \eta_0, \tag{3.26}$$

where $\eta_0, \eta_1, \eta_2, \cdots, \eta_q$ *are constants. Conversely, for arbitrary given* $\eta_0, \eta_1, \eta_2, \cdots, \eta_q$, *the function* F *defined by* (3.26) *belongs to* $\mathfrak{A}(\Gamma)$ *and satisfies* $\mathfrak{S}_h(F) = f$.

It is to be noted here that the additive meromorphic function F defined by (3.26) satisfies as a 0-current the relation

$$dF = 2\pi\theta_f + \sum \eta_\nu A_\nu - 2\pi i Q_f. \tag{3.27}$$

Indeed, putting $T = dF - 2\pi\theta_f - \sum \eta_\nu A_\nu + 2\pi i Q_f$, we infer readily that dT vanishes in the whole of $\mathfrak{M}$ and that T vanishes in $\mathfrak{M} - \Gamma$, while, since, by (3.18), $(1 + iC)dF = -4\pi i Q_f$, we get $(1 + iC)T = 0$, showing that T is a pure 1-current of type 0. Hence, by virtue of Theorem 1.3, T is a holomorphic differential and therefore T must vanish in the whole of $\mathfrak{M}$.

Let $\{\gamma_1, \gamma_2, \cdots, \gamma_{2q}\}$ be a base of 1-cycles on $\mathfrak{M}$ with integral coefficients and $\{A_1, \cdots, A_q\}$ be a base of simple differentials of the first kind such that $(A_\lambda, \bar{A}_\nu) = \delta_{\lambda\nu}$. Again we put

$$\omega_{\nu j} = \gamma_j[A_\nu] = \int_{\gamma_j} A_\nu.$$

$(\omega_{\nu j})$ is called the *period matrix* of the simple differentials of the first kind. Since $HA_\nu = A_\nu$, we have $\omega_{\nu j} = (A_\nu, *H\gamma_j)$. Obviously A_λ can be represented as

$$A_\lambda = \sum_{j=1}^{2q} \alpha_{\lambda j} * H\gamma_j, \tag{3.28}$$

where the coefficients $\alpha_{\lambda j}$ are related with $\omega_{\nu j}$ by the formulae

$$\sum_j \alpha_{\lambda j}\omega_{\nu j} = 0, \qquad \sum_j \bar{\alpha}_{\lambda j}\omega_{\nu j} = \delta_{\lambda\nu}. \tag{3.29}$$

Using (3.21), we infer that the period $\omega_j(F) = \int_{\gamma_j} dF$ of the additive meromorphic function F defined by (3.26) is given by

$$\omega_j(F) = 2\pi i Q_f[H\gamma_j] + \sum_\nu \eta_\nu \omega_{\nu j}. \tag{3.30}$$

Putting

$$\alpha_\lambda(F) = \sum_j \alpha_{\lambda j}\omega_j(F), \qquad \alpha_{\lambda^*}(F) = \sum_j \bar{\alpha}_{\lambda j}\omega_j(F),$$

we introduce the λ- and the λ^*- components $\alpha_\lambda(F)$, $\alpha_{\lambda^*}(F)$ of the "period

vector " $\omega_j(F)$ of F. By means of the " fundamental form " $\Omega = \sum ig_{\alpha\beta^*}dz^\alpha d\bar{z}^\beta$, $*A_\lambda$ can be represented as [38] $*A_\lambda = -i\Omega \cdot A_\lambda$. Hence we have, by (3.28),

$$\sum_{j=1}^{2q} \alpha_{\lambda j} H_{\gamma j} = i\Omega \cdot A_\lambda.$$

Using this formula, (3.29) and the identity $Q_f[\Omega \cdot \bar{A}_\lambda] = 0$, we get from (3.30)

$$\alpha_\lambda(F) = -2\pi Q_f[\Omega \cdot A_\lambda], \qquad \alpha_{\lambda^*}(F) = \eta_\lambda. \tag{3.31}$$

THEOREM 3.4 (EXISTENCE THEOREM). *Let f be an arbitrary function in $\mathfrak{f}(\Gamma, \mathfrak{d})$. Then, there exists on $\mathfrak{M}$ a function $F \in \mathfrak{F}(\Gamma)$ such that $\mathfrak{S}_\mathfrak{h}(F) = f$ if and only if f satisfies*

$$Q_f[\Omega \cdot A_\lambda] = 0, \qquad \text{for } \lambda = 1, 2, \cdots, q, \tag{3.32}$$

and, in case f satisfies (3.32), *F is given by*

$$F(z) = 2\pi \int^z \theta_f + \text{const.} \tag{3.33}$$

Proof. Obviously $F \in \mathfrak{A}(\Gamma)$ belongs to $\mathfrak{F}(\Gamma)$ if and only if $\alpha_\lambda(F) = \alpha_{\lambda^*}(F) = 0$ $(\lambda = 1, 2, \cdots, q)$. Hence, if F belongs to $\mathfrak{F}(\Gamma)$, then, by (3.31), $f = \mathfrak{S}_\mathfrak{h}(F)$ must satisfy (3.32). Conversely, if f satisfies (3.32), then, putting $\eta_\lambda = 0$ in (3.26), we get $F \in \mathfrak{F}(\Gamma)$ satisfying $\mathfrak{S}_\mathfrak{h}(F) = f$, q. e. d.

Since $F \in \mathfrak{F}(\Gamma)$ is determined by $f = \mathfrak{S}_\mathfrak{h}(F)$ uniquely up to an additive constant, $\dim \mathfrak{F}(\Gamma) - 1$ is equal to the number of linearly independent functions f in $\mathfrak{f}(\Gamma, \mathfrak{d})$ satisfying (3.32). Denote by j the number of linearly independent simple differentials A of the first kind satisfying $Q_f[\Omega \cdot A] = 0$, for all $f \in \mathfrak{f}(\Gamma, \mathfrak{d})$. Then (3.32) imposes $q - j$ linearly independent conditions on f. Hence we get

$$\dim\{\Gamma\} = \dim \mathfrak{f}(\Gamma, \mathfrak{d}) - q + j + 1. \tag{3.34}$$

The non-negative integer $\dim \mathfrak{f}(\Gamma, \mathfrak{d}) - \dim\{\Gamma\} + 1$ will be called the *characteristic deficiency* [39] of Γ and be denoted by $chd(\Gamma)$:

$$chd(\Gamma) = \dim \mathfrak{f}(\Gamma, \mathfrak{d}) - \dim\{\Gamma\} + 1.$$

Then, it follows immediately from (3.34) that

$$chd(\Gamma) = q - j; \tag{3.35}$$

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[38] Hodge [12], p. 175.

[39] Goldman [9], § 6. The characteristic deficiency is nothing other than *the deficiency of the characteristic series* of Italian geometers. Cf. Zariski [30], p. 26, pp. 69-71.

thus we see that *the characteristic deficiency of an arbitrary curve is always not greater than q.*

A finite number of simple Picard integrals of the second kind are said to be *independent* if no non-trival linear combination of these integrals reduces to a single valued meromorphic function. By virtue of Theorem 3. 3, we have $\dim \mathfrak{A}(\Gamma) = \dim \mathfrak{f}(\Gamma, \mathfrak{d}) + q + 1$, while the number of independent simple Picard integrals of the second kind which are multiples of $-\Gamma$ is obviously equal to $\dim \mathfrak{A}(\Gamma) - \dim \mathfrak{F}(\Gamma)$. Hence we obtain

THEOREM 3. 5. *The number of independent simple Picard integrals of the second kind which are multiples of* $-\Gamma$ *is equal to* $q + chd(\Gamma)$.

It follows from this theorem that, *if* $\mathfrak{M}$ *contains a curve with the characteristic deficiency q,* $\mathfrak{M}$ *possesses exactly* $2q$ *independent simple Picard integrals of the second kind.*

4. The theorem of Riemann-Roch on reducible curves. By virtue of the formula (3. 34), our problem is reduced to the estimation of $\dim \mathfrak{f}(\Gamma, \mathfrak{d})$. Let Σ be an irreducible curve without singular point. For an arbitrary differential ξ on Σ, we denote by (ξ) the divisor of ξ defined as $(\xi) = \sum_p V_p(\xi) \cdot p$. Now, for an arbitrary divisor $\mathfrak{m}$ on Σ, we denote by $\mathfrak{f}(\mathfrak{m})$ the space of all meromorphic functions on Σ satisfying $(f) + \mathfrak{m} \geqq 0$ and put $\mathfrak{f}^*(\mathfrak{m}) = \mathfrak{f}(\mathfrak{m}) + \mathfrak{f}(0)$ ($\mathfrak{f}(0)$ is the space consisting of all constants). In case $\mathfrak{m} \geqq 0$, $\mathfrak{f}^*(\mathfrak{m})$ coincides with $\mathfrak{f}(\mathfrak{m})$. Again, we denote by $\Xi(\mathfrak{m})$ the space of all differentials ξ on Σ satisfying $(\xi) + \mathfrak{m} \geqq 0$ and put $\Xi^*(\mathfrak{m}) = \Xi(\mathfrak{m}) + \Xi(0)$. $\Xi(0)$ is the space of all differentials of the first kind and therefore $\dim \Xi(0)$ is equal to the genus π of Σ.

Let $\mathfrak{a} = \sum m_p p$, $\mathfrak{b} = \sum n_q q$ be two *positive* divisors on Σ such that the carriers $|\mathfrak{a}|$ and $|\mathfrak{b}|$ have no point in common. Now, for arbitrary $\xi \in \Xi(\mathfrak{b})$ and $f \in \mathfrak{f}(\mathfrak{a})$, we define the product (ξf) as

$$(\xi f) = \sum_{q \in |\mathfrak{b}|} \mathrm{Res}_q[f\xi]\,; \text{ hence } (\xi f) = -\sum_{p \in |\mathfrak{a}|} \mathrm{Res}_p[f\xi].$$

As one readily infers, (ξf) *vanishes for all* $\xi \in \Xi(\mathfrak{b})$ *if and only if* $f \in \mathfrak{f}^*(\mathfrak{a} - \mathfrak{b})$. Again, if $\xi \in \Xi^*(\mathfrak{b} - \mathfrak{a})$, *then* (ξf) *vanishes for all* $f \in \mathfrak{f}(\mathfrak{a})$. Thus, in the product (ξf), ξ and f can be considered as the elements of the factor spaces $\bar{\Xi} = \Xi(\mathfrak{b})/\Xi^*(\mathfrak{b} - \mathfrak{a})$ and $\bar{\mathfrak{f}} = \mathfrak{f}(\mathfrak{a})/\mathfrak{f}^*(\mathfrak{a} - \mathfrak{b})$, respectively. Now we shall show that ξ *belongs to* $\Xi^*(\mathfrak{b} - \mathfrak{a})$ *if* (ξf) *vanishes for all* $f \in \mathfrak{f}(\mathfrak{a})$ so that $\bar{\Xi}$ *and* $\bar{\mathfrak{f}}$ *are dual to each other with respect to the product* (ξf). To prove this, it is sufficient to verify that $\dim \bar{\Xi} = \dim \bar{\mathfrak{f}}$. By virtue of the theorem of Riemann-Roch, we have

$$\dim \mathfrak{f}(\mathfrak{a}) = \deg \mathfrak{a} - \pi + i(\mathfrak{a}) + 1,$$

$$\dim \mathfrak{f}^*(\mathfrak{a} - \mathfrak{b}) = \deg \mathfrak{a} - \deg \mathfrak{b} - \pi + i(\mathfrak{a} - \mathfrak{b}) + 2,$$

where π is the genus of Σ and $i(\mathfrak{a})$, $i(\mathfrak{a} - \mathfrak{b})$ are the indices of specialties of of $\mathfrak{a}$, $\mathfrak{a} - \mathfrak{b}$, respectively. Hence we get

$$\dim \bar{\mathfrak{f}} = \deg \mathfrak{b} - 1 + i(\mathfrak{a}) - i(\mathfrak{a} - \mathfrak{b}).$$

On the other hand, since

$$i(\mathfrak{a}) = \dim \Xi(-\mathfrak{a}) = \dim [\Xi(\mathfrak{b} - \mathfrak{a}) \cap \Xi(0)],\ i(\mathfrak{a} - \mathfrak{b}) = \dim \Xi(\mathfrak{b} - \mathfrak{a}),$$

we have

$$\dim \Xi^*(\mathfrak{b} - \mathfrak{a}) = i(\mathfrak{a} - \mathfrak{b}) + \pi - i(\mathfrak{a}),$$

whence we get, using $\dim \Xi(\mathfrak{b}) = \deg \mathfrak{b} - 1 + \pi$,

$$\dim \bar{\Xi} = \deg \mathfrak{b} - 1 + i(\mathfrak{a}) - i(\mathfrak{a} - \mathfrak{b}).$$

Hence we obtain $\dim \bar{\Xi} = \dim \bar{\mathfrak{f}}$, q. e. d.

Now we turn to the reducible curve $\Gamma = \sum_{\nu=1}^{\mu} \Gamma_\nu$ on $\mathfrak{M}$. For an arbitrary divisor $\mathfrak{m}$ on the non-singular model $\mathbf{\Gamma}$ of Γ, we denote by $\mathfrak{f}(\mathfrak{m})$ the space of all meromorphic functions f on $\mathbf{\Gamma}$ satisfying $(f) + \mathfrak{m} \geqq 0$, by $\Xi(\mathfrak{m})$ the space of all differentials ξ on $\mathbf{\Gamma}$ satisfying $(\xi) + \mathfrak{m} \geqq 0$ and put $\mathfrak{f}^*(\mathfrak{m}) = \mathfrak{f}(\mathfrak{m}) + \mathfrak{f}(0)$, $\Xi^*(\mathfrak{m}) = \Xi(\mathfrak{m}) + \Xi(0)$. We choose the characteristic divisor $\mathfrak{d}$ of Γ so that $|\mathfrak{d}|$ and $|\mathfrak{c}|$ have no point in common and decompose $\mathfrak{d}$ as follows:

$$\mathfrak{d} = \mathfrak{d}^+ - \mathfrak{d}^-, \quad \mathfrak{d}^+ \geqq 0,\ \mathfrak{d}^- \geqq 0 \text{ and } |\mathfrak{d}^+| \cap |\mathfrak{d}^-| \text{ is empty.}$$

Again, we put $\mathfrak{a} = \mathfrak{d}^+$, $\mathfrak{b} = \mathfrak{d}^- + \mathfrak{c}$ and introduce the product

$$(\xi f) = \sum_{q \in |\mathfrak{b}|} \mathrm{Res}_q[f\xi]$$

of arbitrary $\xi \in \Xi(\mathfrak{b})$ and $f \in \mathfrak{f}(\mathfrak{a})$. The non-singular model $\mathbf{\Gamma}$ is the direct sum: $\mathbf{\Gamma} = \sum_\nu \mathbf{\Gamma}_\nu$ of irreducible components $\mathbf{\Gamma}_\nu$. Corresponding to this, $\mathfrak{a}$, $\mathfrak{b}$, ξ, f can be decomposed into the direct sums: $\mathfrak{a} = \sum_\nu \mathfrak{a}_\nu$, $\mathfrak{b} = \sum_\nu \mathfrak{b}_\nu$, $\xi = \sum_\nu \xi_\nu$, $f = \sum_\nu f_\nu$ of $\mathfrak{a}_\nu$, $\mathfrak{b}_\nu$, ξ_ν, f_ν defined on $\mathbf{\Gamma}_\nu$ so that we have

$$(\xi f) = \sum_\nu (\xi_\nu f_\nu), \qquad \xi_\nu \in \Xi(\mathfrak{b}_\nu),\ f_\nu \in \mathfrak{f}(\mathfrak{a}_\nu),$$

where $(\xi_\nu f_\nu) = \sum_{q \in |\mathfrak{b}_\nu|} \mathrm{Res}_q[f_\nu \xi_\nu]$. Hence, applying the "duality theorem" proved above to each pair of divisors $\mathfrak{a}_\nu$, $\mathfrak{b}_\nu$, we see immediately that the

factor spaces $\Xi(\mathfrak{d}^- + \mathfrak{c})/\Xi^*(\mathfrak{c} - \mathfrak{d})$ and $\mathfrak{f}(\mathfrak{d}^+)/\mathfrak{f}^*(\mathfrak{d} - \mathfrak{c})$ *are dual to each other with respect to the product* (ξf).

It is convenient for our present purpose to introduce the following notations: Let $\mathfrak{m} = \sum m_p p$ be a divisor on $\mathbf{\Gamma}$ and $\{\xi_p | p \in \mathfrak{T}\}$ be a system of differentials ξ_p defined respectively in a neighborhood U_p of $p \in \mathfrak{T}$, where $\mathfrak{T}$ is a finite set of points. Again, let ξ be a differential on $\mathbf{\Gamma}$. Then, we shall say that ξ is congruent to $\sum_{p \in \mathfrak{T}} \xi_p$ modulo $\mathfrak{m}$ and write

$$\xi \equiv \sum_{p \in \mathfrak{T}} \xi_p \quad (\mathfrak{m}), \tag{4.1}$$

if ξ satisfies

$$V_p(\xi - \xi_p) \geqq m_p, \text{ for } p \in \mathfrak{T}; \qquad V_p(\xi) \geqq m_p, \text{ for } p \notin \mathfrak{T}.$$

The *formal sum* $\sum \xi_p$ in the right-hand side of (4.1) stands simply for the *system* or the *vector* $\{\xi_p\}$. Again, a finite number of formal sums $\sum_p \xi_p^{(\nu)}$ $(\nu = 1, 2, \cdots)$ will be called *independent modulo* $\mathfrak{m}$, if no non-trivial linear combination $\sum a_\nu \sum_p \xi_p^{(\nu)}$ satisfies

$$0 \equiv \sum a_\nu \sum_p \xi_p^{(\nu)} \quad (\mathfrak{m}).$$

For each singular point $\mathfrak{p}$ of Γ, we introduce the formal sums

$$\sigma_{\mathfrak{p}}^{(k,l)} = \sum_{\phi(p)=\mathfrak{p}} (\phi_p^{\,1})^k (\phi_p^{\,2})^l \sigma_p, \qquad (k, l = 0, 1, 2, \cdots)$$

and denote by $l_{\mathfrak{p}}$ the number of $\sigma_{\mathfrak{p}}^{(k,l)}$ which are independent modulo 0, where 0 denotes the divisor *zero*: $0 = \sum_p 0 \cdot p$. If one considers

$$(\beta_{\mathfrak{p}}) \qquad \sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[(\phi_p^{\,1})^k (\phi_p^{\,2})^l f h_p \sigma_p] = 0 \qquad (k, l = 0, 1, 2, \cdots)$$

as the conditions for the system $\{f h_p | p \in \phi^{-1}(\mathfrak{p})\}$, then $l_{\mathfrak{p}}$ represents the number of linearly independent conditions involved in $(\beta_{\mathfrak{p}})$. For any simple point $\mathfrak{p}$, we put $l_{\mathfrak{p}} = 0$.

Now, let $\Xi(\Gamma, -\mathfrak{d})$ be the space consisting of all differentials ξ on $\mathbf{\Gamma}$ which are congruent modulo $\mathfrak{d}$ to some linear combinations of the formaal sums $\sum_{\phi(p)=\mathfrak{p}} (\phi_p^{\,1})^k (\phi_p^{\,2})^l h_p \sigma_p$, i. e.

$$\Xi(\Gamma, -\mathfrak{d}) = \{\xi | \xi \equiv \sum_{\mathfrak{p}} \sum_{k,l} a_{\mathfrak{p}kl} \sum_{\phi(p)=\mathfrak{p}} (\phi_p^{\,1})^k (\phi_p^{\,2})^l h_p \sigma_p \quad (\mathfrak{d})\},$$

where $a_{\mathfrak{p}kl}$ $(k, l = 0, 1, 2, \cdots)$ are arbitrary constants and $\sum_{\mathfrak{p}}$ is the sum extended over all singular points of Γ. Then we have

THEOREM 4.1. *The dimension of* $\mathfrak{f}(\Gamma, \mathfrak{d})$ *is given by the formula*

$$\dim \mathfrak{f}(\Gamma, \mathfrak{d}) = \deg \mathfrak{d} - \sum_{\nu=1}^{\mu} \pi_\nu + \mu - \sum_{\mathfrak{p}} l_{\mathfrak{p}} + i, \tag{4.2}$$

where μ is the number of irreducible components of $\Gamma = \sum_{\nu=1}^{\mu} \Gamma_\nu$, π_ν is the genus of Γ_ν and $i = \dim \Xi(\Gamma, -\mathfrak{d})$.

Proof. Assume first that $|\mathfrak{d}|$ and $|\mathfrak{c}|$ have no point in common and that each component $\mathfrak{d}^-_\nu$ of $\mathfrak{d}^-$ is *positive*. Then, for every $\sum_{\phi(p)=\mathfrak{p}} (\phi_p^{\,1})^k (\phi_p^{\,2})^l h_p \sigma_p$, there exists at least one differential ξ on Γ satisfying

$$\xi \equiv \sum_{\phi(p)=\mathfrak{p}} (\phi_p^{\,1})^k (\phi_p^{\,2})^l h_p \sigma_p \quad (-\mathfrak{d}^-).$$

Hence, denoting by Ξ the space consisting of all differentials ξ satisfying

$$\xi \equiv \sum_{\mathfrak{p}} \sum_{k,l} a_{\mathfrak{p}kl} \sum_{\phi(p)=\mathfrak{p}} (\phi_p^{\,1})^k (\phi_p^{\,2})^l h_p \sigma_p \quad (-\mathfrak{d}^-),$$

$a_{\mathfrak{p}kl}$ being arbitrary constants, we infer readily that the space $\mathfrak{f}^*(\Gamma, \mathfrak{d}) = \mathfrak{f}(\Gamma, \mathfrak{d}) + \mathfrak{f}(0)$ is the subspace of $\mathfrak{f}(\mathfrak{d}^+)$ consisting of all $f \in \mathfrak{f}(\mathfrak{d}^+)$ satisfying $(\xi f) = 0$, for all $\xi \in \Xi$; thus $\mathfrak{f}^*(\Gamma, \mathfrak{d})$ is characterized as the space which is orthogonal to Ξ in the dualism between $\mathfrak{f}(\mathfrak{d}^+)/\mathfrak{f}^*(\mathfrak{d} - \mathfrak{c})$ and $\Xi(\mathfrak{d}^- + \mathfrak{c})/\Xi^*(\mathfrak{c} - \mathfrak{d})$ with respect to the product (ξf). Obviously Ξ contains $\Xi(0)$ but does not necessarily contain $\Xi(\mathfrak{c} - \mathfrak{d})$. The space which is orthogonal to $\mathfrak{f}^*(\Gamma, \mathfrak{d})$ is therefore $\Xi + \Xi(\mathfrak{c} - \mathfrak{d})$. Hence we get

$$\dim \mathfrak{f}^*(\Gamma, \mathfrak{d}) - \dim \mathfrak{f}^*(\mathfrak{d} - \mathfrak{c}) = \dim \Xi(\mathfrak{d}^- + \mathfrak{c}) - \dim[\Xi + \Xi(\mathfrak{c} - \mathfrak{d})]. \tag{4.3}$$

Obviously we have

$$\dim \mathfrak{f}^*(\Gamma, \mathfrak{d}) = \dim \mathfrak{f}(\Gamma, \mathfrak{d}) + \mu,$$

$$\dim \mathfrak{f}^*(\mathfrak{d} - \mathfrak{c}) = \dim \mathfrak{f}(\mathfrak{d} - \mathfrak{c}) + \mu,$$

$$\dim \Xi(\mathfrak{d}^- + \mathfrak{c}) = \deg \mathfrak{c} + \dim \Xi(\mathfrak{d}^-)$$

and, since $\Xi \cap \Xi(\mathfrak{c} - \mathfrak{d}) = \Xi(\Gamma, -\mathfrak{d})$,

$$\dim [\Xi + \Xi(\mathfrak{c} - \mathfrak{d})] = \dim \Xi + \dim \Xi(\mathfrak{c} - \mathfrak{d}) - i,$$

while, applying the theorem of Riemann-Roch to each Γ_ν, we get

$$\dim \mathfrak{f}(\mathfrak{d} - \mathfrak{c}) = \deg \mathfrak{d} - \deg \mathfrak{c} - \sum \pi_\nu + \mu + \dim \Xi(\mathfrak{c} - \mathfrak{d}).$$

Inserting these formulae in (4.3) and using $\dim \Xi - \dim \Xi(\mathfrak{d}^-) = \sum_{\mathfrak{p}} l_{\mathfrak{p}}$, we obtain

$$\dim \mathfrak{f}(\Gamma, \mathfrak{d}) = \deg \mathfrak{d} - \sum \pi_\nu + \mu - \sum_{\mathfrak{p}} l_{\mathfrak{p}} + i.$$

Thus (4.2) is proved in the case that $\mathfrak{d}$ is subject to the conditions: $\mathfrak{d}^-_\nu > 0$,

$|\mathfrak{d}| \cap |\mathfrak{c}|$ is empty. Now, consider an arbitrary characteristic divisor $\mathfrak{d}'$ not satisfying the above conditions. $\mathfrak{d}'$ is determined by a system $\{h'_p\}$ which is obtained from $\{h_p\}$ by the transformation $h_p \to h'_p = g \cdot h_p$, g being a meromorphic function on $\mathbf{\Gamma}$. Then, as one readily infers, $\mathfrak{f}(\Gamma, \mathfrak{d}')$ or $\Xi(\Gamma, -\mathfrak{d}')$ is obtained respectively from $\mathfrak{f}(\Gamma, \mathfrak{d})$ or $\Xi(\Gamma, -\mathfrak{d})$ by the transformation $f \to f' = g^{-1} \cdot f$ or $\xi \to \xi' = g \cdot \xi$, so that we have

$$\dim \mathfrak{f}(\Gamma, \mathfrak{d}') = \dim \mathfrak{f}(\Gamma, \mathfrak{d}), \qquad \dim \Xi(\Gamma, -\mathfrak{d}') = \dim \Xi(\Gamma, -\mathfrak{d}).$$

Hence (4. 2) is valid also for $\mathfrak{d}'$, q. e. d.

5. The theorem of Riemann-Roch on compact analytic surfaces. Let $D = \sum m_\nu \Gamma_\nu$ be the representation of an arbitrary divisor D on $\mathfrak{M}$ in terms of the irreducible components Γ_ν of D. Then, at each point $\mathfrak{p} \in \mathfrak{M}$, D can be *represented* by the meromorphic function

$$R_\mathfrak{p}(D; z^1, z^2) = \prod_\nu \{R_{\nu\mathfrak{p}}(z^1, z^2)\}^{m_\nu},$$

where each $R_{\nu\mathfrak{p}}(z^1, z^2) = 0$ is the *minimal* local equation of Γ_ν at $\mathfrak{p}$. Put

$$h_p(D; \tau_p) = R_{\phi(p)}(D; \phi_p{}^1(\tau_p), \phi_p{}^2(\tau_p)).$$

Then $V_p(h_p(D))$ gives the intersection number of D and the branch Γ_p at $\mathfrak{p} = \phi(p)$. Putting $D \cdot \Gamma = \sum_p V_p(h_p(D)) \cdot p$, we introduce the *intersection divisor* $D \cdot \Gamma$ *of* D *and* Γ on $\mathbf{\Gamma}$. Obviously we have

$$I(D, \Gamma) = \deg (D \cdot \Gamma). \tag{5. 1}$$

Now, inserting (4. 2) in (3. 34) and using $I(\Gamma, \Gamma) = \deg \mathfrak{d}$, we get immediately

$$\dim \{\Gamma\} = I(\Gamma, \Gamma) - \sum_{\nu=1}^{\mu} \pi_\nu + \mu - \sum_\mathfrak{p} l_\mathfrak{p} - q + i + j + 1.$$

Putting $\pi_\Gamma = \sum_{\nu=1}^{\mu} \pi_\nu - \mu + 1 + \frac{1}{2} \sum_p c_p$, we obtain therefore

$$\dim \{\Gamma\} = I(\Gamma, \Gamma) - \pi_\Gamma - q + \tfrac{1}{2} \sum_p c_p - \sum_\mathfrak{p} l_\mathfrak{p} + i + j + 2. \tag{5. 2}$$

The quantity π_Γ is called the *virtual genus*[40] of Γ. Now we shall show that *the virtual genus* π_Γ *of* Γ *is determined uniquely by the homology class of* Γ.[41] For this purpose, we first introduce the notion of the *canonical cycles*.[42] Let

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[40] Cf. Zariski [30], p. 30.

[41] This fact was suggested by A. Weil in his letter to the author.

[42] Weil [25].

$\{\mathfrak{U}_\lambda\}$ be a finite covering of $\mathfrak{M}$ by the "spherical" domains $\mathfrak{U}_\lambda$ each of which is covered by a single system of local coordinates $(z_\lambda^1, z_\lambda^2)$ such that each intersection $\mathfrak{U}_\lambda \cap \mathfrak{U}_\nu$ is simply connected (if it is not empty). Then, putting $dz_\nu^1 dz_\nu^2 = M_{\lambda\nu} dz_\lambda^1 dz_\lambda^2$, in $\mathfrak{U}_\lambda \cap \mathfrak{U}_\nu$, we get a system $\{M_{\lambda\nu}\}$ of the non-vanishing holomorphic functions $M_{\lambda\nu}$ defined respectively in $\mathfrak{U}_\lambda \cap \mathfrak{U}_\nu$ satisfying

$$M_{\lambda\nu} M_{\nu\lambda} = 1, \text{ in } \mathfrak{U}_\lambda \cap \mathfrak{U}_\nu; \qquad M_{\lambda\mu} M_{\mu\nu} M_{\nu\lambda} = 1, \text{ in } \mathfrak{U}_\lambda \cap \mathfrak{U}_\mu \cap \mathfrak{U}_\nu.$$

This system $\{M_{\lambda\nu}\}$ determines a *fibre bundle* $\mathfrak{F}$ over $\mathfrak{M}$ in the following manner: Consider the system $\{\mathfrak{U}_\lambda \times \mathfrak{E}\}$ of the direct products $\mathfrak{U}_\lambda \times \mathfrak{E}$ of $\mathfrak{U}_\lambda$ and the *projective line* $\mathfrak{E}$ and identify two points $\mathfrak{p} \times \zeta_\lambda \varepsilon\, \mathfrak{U}_\lambda \times \mathfrak{E}$ and $\mathfrak{p} \times \zeta_\nu \varepsilon\, \mathfrak{U}_\nu \times \mathfrak{E}$ if $\mathfrak{p} \varepsilon\, \mathfrak{U}_\lambda \cap \mathfrak{U}_\nu$ and $\zeta_\lambda/\zeta_\nu = M_{\lambda\nu}(\mathfrak{p})$. Then we get from $\{\mathfrak{U}_\lambda \times \mathfrak{E}\}$ a compact analytic manifold $\mathfrak{F}$ consisting of all points $\mathfrak{p} \times \zeta_\lambda$ which is obviously a fibre bundle over the base space $\mathfrak{M}$; the fibre of $\mathfrak{F}$ is $\mathfrak{E}$ and the transformation group in the fibre $\mathfrak{E}$ is the multiplicative group of complex numbers. Furthermore it can be easily verified that the analytic structure of $\mathfrak{F}$ relative to $\mathfrak{M}$ is determined uniquely by $\mathfrak{M}$. For an arbitrary point $P = \mathfrak{p} \times \zeta_\lambda$ of $\mathfrak{F}$, we denote by $\pi(P)$ the projection $\mathfrak{p}$ of P on the base space $\mathfrak{M}$. The subsets $\mathfrak{M}_0 = \{\mathfrak{p} \times 0 \,|\, \mathfrak{p} \varepsilon\, \mathfrak{M}\}$ and $\mathfrak{M}_\infty = \{\mathfrak{p} \times \infty \,|\, \mathfrak{p} \varepsilon\, \mathfrak{M}\}$ of $\mathfrak{F}$ are compact analytic sub-manifolds of complex dimension 2. Now, each $d \log M_{\lambda\nu}$ is a holomorphic simple differential in $\mathfrak{U}_\lambda \cap \mathfrak{U}_\nu$ and we have

$$d \log M_{\lambda\nu} + d \log M_{\nu\lambda} = 0, \quad \text{in } \mathfrak{U}_\lambda \cap \mathfrak{U}_\nu,$$

$$d \log M_{\lambda\mu} + d \log M_{\mu\nu} + d \log M_{\nu\lambda} = 0, \quad \text{in } \mathfrak{U}_\lambda \cap \mathfrak{U}_\mu \cap \mathfrak{U}_\nu.$$

Hence we can find a system $\{\Psi_\lambda\}$ of the 1-forms Ψ_λ of class C^∞ defined respectively in $\mathfrak{U}_\lambda$ such that [43]

$$\Psi_\lambda - \Psi_\nu = (1/2\pi i)\, d \log M_{\lambda\nu}, \quad \text{in } \mathfrak{U}_\lambda \cap \mathfrak{U}_\nu.$$

Putting $\mathrm{X} = d\Psi_\lambda$, we get a well defined 2-form X of class C^∞ on $\mathfrak{M}$, since $d\Psi_\lambda = d\Psi_\nu$ in $\mathfrak{U}_\lambda \cap \mathfrak{U}_\nu$. Considering $M_{\lambda\nu}(\mathfrak{p}) = M_{\lambda\nu}(\pi(P))$ as a function of $P = \mathfrak{p} \times \zeta_\lambda = \mathfrak{p} \times \zeta_\nu$ defined in $\pi^{-1}(\mathfrak{U}_\lambda \cap \mathfrak{U}_\nu)$, we have the identity $M_{\lambda\nu} = \zeta_\lambda/\zeta_\nu$. Consequently we get $d \log M_{\lambda\nu} = d \log \zeta_\lambda - d \log \zeta_\nu$ and therefore

$$\Psi_\lambda - (1/2\pi i)\, d \log \zeta_\lambda = \Psi_\nu - (1/2\pi i)\, d \log \zeta_\nu, \quad \text{in } \pi^{-1}(\mathfrak{U}_\lambda \cap \mathfrak{U}_\nu),$$

where Ψ_λ, Ψ_ν are to be considered as 1-forms defined on $\pi^{-1}(\mathfrak{U}_\lambda)$, $\pi^{-1}(\mathfrak{U}_\nu)$, respectively. This result shows that $\Psi = \Psi_\lambda - (1/2\pi i)\, d \log \zeta_\lambda$ is a well defined 1-current on $\mathfrak{F}$. Now, we have $d \cdot d \log \zeta_\lambda = 2\pi i (\mathfrak{M}_0 - \mathfrak{M}_\infty)$, in

[43] Weil [24], p. 113.

$\pi^{-1}(\mathfrak{U}_\lambda)$. In fact, for an arbitrary 4-form Φ of class C^∞ with $|\Phi| \subset \pi^{-1}(\mathfrak{U}_\lambda)$, we get, putting $\zeta = \zeta_\lambda$ and $\eta = 1/\zeta_\lambda$,

$$d \cdot d \log \zeta[\Phi] = d \log \zeta[d\Phi] = \int (d\zeta/\zeta) \cdot d\Phi$$

$$= -\int d\{(d\zeta/\zeta) \cdot \Phi\} = -\lim_{\epsilon \to +0} \int_{\epsilon < |\xi| < 1/\epsilon} d\{(d\zeta/\zeta) \cdot \Phi\}$$

$$= \lim_{\epsilon \to +0} \{\int_{|\xi|=\epsilon} (d\zeta/\zeta) \cdot \Phi - \int_{|\eta|=\epsilon} (d\eta/\eta) \cdot \Phi\}$$

$$= 2\pi i \{\int_{\mathfrak{M}_0} \Phi - \int_{\mathfrak{M}_\infty} \Phi\} = 2\pi i (\mathfrak{M}_0 - \mathfrak{M}_\infty)[\Phi],$$

proving the above formula. Hence we obtain

$$\mathrm{X} = d\Psi + \mathfrak{M}_0 - \mathfrak{M}_\infty, \tag{5.3}$$

where X is to be considered as a 2-form on $\mathfrak{F}$. The formula (5.3) shows that *the cohomology class of the 2-form* X *on* $\mathfrak{M}$ *is uniquely determined by the analytic structure of* $\mathfrak{M}$. Moreover it can be shown that the cohomology class of X is the *second basic characteristic class*[44] of the manifold $\mathfrak{M}$. *By a canonical cycle on* $\mathfrak{M}$ *we shall mean a 2-cycle* $\mathfrak{k}$ *such that* $I(\mathfrak{k}, Z) = Z[\mathrm{X}]$ *for all 2-cycles* Z *on* $\mathfrak{M}$.

Now we shall prove the formula

$$2\pi_\Gamma - 2 = I(\Gamma, \Gamma) + I(\mathfrak{k}, \Gamma). \tag{5.4}$$

For this purpose, we choose for each point $p \in \mathbf{\Gamma}$ a sufficiently small neighborhood U_p of p so that $\phi(U_p)$ is contained in one of the domains $\mathfrak{U}_\lambda$ which will be denoted by $\mathfrak{U}_{\lambda(p)}$: $\phi(U_p) \subset \mathfrak{U}_{\lambda(p)}$. Then, putting

$$m_{pq}(\tau) = M_{\lambda(p)\lambda(q)}(\phi(\tau)), \quad \text{in } U_p \cap U_q,$$

we get a system $\{m_{pq}\}$ of the holomorphic functions $\{m_{pq}\}$ satisfying the conditions

$$m_{pq}m_{qp} = 1, \quad \text{in } U_p \cap U_q; \qquad m_{pq}m_{qr}m_{rp} = 1, \quad \text{in } U_p \cap U_q \cap U_r.$$

Hence, by the same method as in the proof of Theorem 2.1, we can show that there exists a system $\{k_p\}$ of meromorphic functions k_p defined respectively in U_p satisfying $k_p/k_q = m_{pq}$, in $U_p \cap U_q$. Now, we associate with each point τ in $U_p \subset \mathbf{\Gamma}$ the point $P(\tau) = \phi(\tau) \times k_p(\tau)$ in $\mathfrak{U}_{\lambda(p)} \times \mathfrak{E}$. Then, since

$$k_p(\tau)/k_q(\tau) = m_{pq}(\tau) = M_{\lambda(p)\lambda(q)}(\phi(\tau)), \quad \text{in } U_p \cap U_q,$$

[44] See Chern [6], Theorem 8.

we have $\phi(\tau)\times k_p(\tau) = \phi(\tau)\times k_q(\tau)$, for $\tau \in U_p \cap U_q$. This shows that $\tau \to P(\tau) = \phi(\tau)\times k_p(\tau)$ is a well defined mapping which maps $\mathbf{\Gamma}$ onto a compact analytic curve $P(\mathbf{\Gamma})$ in $\mathfrak{F}$. Obviously we have $\pi(P(\tau)) = \phi(\tau)$ and therefore $\pi(P(\mathbf{\Gamma})) = \Gamma$. Hence we get, using (5.3),

$$\Gamma[X] = P(\mathbf{\Gamma})[X] = (*X, P(\Gamma)) = (*\mathfrak{M}_0 - *\mathfrak{M}_\infty, P(\mathbf{\Gamma})),$$

yielding $I(\mathfrak{k}, \Gamma) = \Gamma[X] = I(\mathfrak{M}_0 - \mathfrak{M}_\infty, P(\mathbf{\Gamma}))$. It can easily be verified that

$$I(\mathfrak{M}_0 - \mathfrak{M}_\infty, P(\mathbf{\Gamma})) = \sum_p V_p(k_p), \tag{5.5}$$

where V_p denotes the valuation with the center p. In fact, a point $P(p)$ on $P(\mathbf{\Gamma})$ meets with $\mathfrak{M}_0$ if and only if $k_p(p) = 0$ (we write p instead of τ). The intersection number $I(\mathfrak{M}_0, P(\mathbf{\Gamma}))$ is therefore equal to the sum: $\sum_p I(\mathfrak{M}_0, P(U_p))$ extended over all p such that $k_p(p) = 0$, while we have

$$I(\mathfrak{M}_0, P(U_p)) = (1/2\pi i) \oint_p d \log k_p(\tau_p) = V_p(k_p).$$

Similarly, $P(p)$ meets with $\mathfrak{M}_\infty$ if and only if $1/k_p(p) = 0$ and, if $1/k_p(p) = 0$, we have

$$I(\mathfrak{M}_\infty, P(U_p)) = (1/2\pi i) \oint_p d \log(1/k_p(\tau_p)) = -V_p(k_p).$$

Hence we get (5.5). Consequently we obtain

$$I(\mathfrak{k}, \Gamma) = \sum_p V_p(k_p). \tag{5.6}$$

The differential σ_p depends on the choice of the local coordinates z^1, z^2 in $\mathfrak{U}_{\phi(p)}$. Using ${z_{\lambda(p)}}^1$, ${z_{\lambda(p)}}^2$ as the local coordinates in $\mathfrak{U}_{\phi(p)}$, we have by (3.17)

$$k_q h_q \sigma_q = k_q [M_{\lambda(p)\lambda(q)}]_{z=\phi(\tau)} h_p \sigma_p = k_q m_{pq} h_p \sigma_p = k_p h_p \sigma_p, \quad \text{in } U_p \cap U_q.$$

This proves that $\xi = k_p h_p \sigma_p$ is a well defined differential on $\mathbf{\Gamma}$. The divisor (ξ) of ξ is represented as

$$(\xi) = \sum_p V_p(k_p) p + \mathfrak{d} - \mathfrak{c},$$

while we have

$$\deg(\xi) = \sum_{\nu=1}^{\mu} (2\pi_\nu - 2) = 2\Big(\sum_{\nu=1}^{\mu} \pi_\nu - \mu\Big).$$

Hence we get

$$2\pi_\Gamma - 2 = \deg(\xi) + \deg \mathfrak{c} = \deg \mathfrak{d} + \sum_p V_p(k_p).$$

Combined with (5.6), this yields immediately (5.4). The formula (5.4) shows clearly that π_Γ is determined uniquely by the homology class of Γ.

By a *canonical divisor* on $\mathfrak{M}$ we shall mean the divisor K of an arbitrary double differential $W = W_{12}(z^1, z^2)dz^1dz^2$ (defined in the whole of $\mathfrak{M}$) determined by means of the relation: $W_{12}(z^1, z^2) = R_{\mathfrak{p}}(K; z^1, z^2)$. Obviously any two canonical divisors K, K' are linearly equivalent to each other and therefore *the divisor class* $\{K\}$ *of a canonical divisor* K *is determined uniquely by* $\mathfrak{M}$, *if at least one double differential* $W \neq 0$ *ever exists on* $\mathfrak{M}$. Incidentally, the divisor of a double differential W will be denoted by (W). Now, assuming the existence of a double differential $W \neq 0$, we shall show that *any canonical divisor* $K = (W)$ *is a canonical cycle.* Let $\{\mathfrak{U}_\lambda\}$, $z_\lambda{}^a$, $M_{\lambda\nu}$, etc. have the same meaning as above. Then, putting $W = R_\lambda(K; z_\lambda{}^1, z_\lambda{}^2)dz_\lambda{}^1dz_\lambda{}^2$, in $\mathfrak{U}_\lambda$, we have $M_{\lambda\nu} = R_\lambda(K)/R_\nu(K)$ in $\mathfrak{U}_\lambda \cap \mathfrak{U}_\nu$ and therefore

$$\Psi_\lambda - \Psi_\nu = (1/2\pi i)[d \log R_\lambda(K) - d \log R_\nu(K)], \quad \text{in } \mathfrak{U}_\lambda \cap \mathfrak{U}_\nu.$$

This shows that $\Phi = \Psi_\lambda - (1/2\pi i)d \log R_\lambda(K)$ is a well defined 1-form of class C^∞ in $\mathfrak{M} - |K|$. Now, let $Z = \sum c_j T_j$ be an arbitrary 2-cycle on $\mathfrak{M}$ represented as a linear combination of a finite number of differentiable simplices T_j such that each T_j is covered by a gingle neighborhood $\mathfrak{U}_{\lambda(j)}$ and that each $|dT_j|$ does not meet with $|K|$. Then we have

$$\begin{aligned} Z[\mathrm{X}] &= \sum c_j T_j[\mathrm{X}] = \sum c_j T_j[d\Psi_{\lambda(j)}] = -\sum c_j dT_j[\Psi_{\lambda(j)}] \\ &= -\sum c_j dT_j[\Phi] - (1/2\pi i)\sum c_j dT_j[d \log R_{\lambda(j)}(K)], \end{aligned}$$

while $\sum c_j dT_j = dZ = 0$ and, as (2.14) shows, $(1/2\pi i)dT_j[d \log R_{\lambda(j)}(K)] = -I(T_j, K)$. Hence we get $Z[\mathrm{X}] = I(K, Z)$, proving that K is a canonical cycle on $\mathfrak{M}$.

It follows from (5.4) that, by means of the canonical divisor K, the virtual genus π_Γ of Γ is represented as

$$2\pi_\Gamma - 2 = I(\Gamma, \Gamma) + I(K, \Gamma). \tag{5.7}$$

Now we shall prove the identity [45]

$$l_{\mathfrak{p}} = \tfrac{1}{2} \sum_{\phi(p)=\mathfrak{p}} c_p. \tag{5.8}$$

For that purpose, we first deduce from Theorem 3.2 the inequality

$$l_{\mathfrak{p}} \leqq \tfrac{1}{2} \sum_{\phi(p)=\mathfrak{p}} c_p. \tag{5.9}$$

Let $\mathfrak{p}$ be a fixed singular point of Γ. We denote by u a *vector* (or a *system*) $\{u_p | p \in \phi^{-1}(\mathfrak{p})\}$ consisting of polynomials $u_p = u_p(\tau_p)$ in τ_p of degree $\leqq c_p - 1$

[45] An arithmetic proof of this identity based on local considerations was given recently by D. Gorenstein. See Gorenstein [10].

and by $\mathfrak{l}_\mathfrak{p}$ the linear space consisting of all such vectors $u = \{u_p\}$. Obviously we have

$$\dim \mathfrak{l}_\mathfrak{p} = \sum_{\phi(p)=\mathfrak{p}} c_p.$$

Let (z^1, z^2) be a system of local coordinates with the origin $\mathfrak{p}$. Then every monomial $(z^1)^k(z^2)^l$ induces a holomorphic function $(\phi_p{}^1)^k(\phi_p{}^2)^l$ on each branch Γ_p, $p \in \phi^{-1}(\mathfrak{p})$. Denoting by $u_p{}^{(k,l)}$ the polynomial obtained from $(\phi_p{}^1)^k(\phi_p{}^2)^l$ by reducing it modulo $\tau_p{}^{c_p}$, we define the vectors $u^{(k,l)} = \{u_p{}^{(k,l)}\}$, $(k, l = 0, 1, 2, \cdots)$. Let $\mathfrak{L}_\mathfrak{p}$ be the subspace of $\mathfrak{l}_\mathfrak{p}$ generated by these induced vectors $u^{(k,l)}$, $k, l = 0, 1, 2, \cdots$. Then we have $\dim \mathfrak{L}_\mathfrak{p} = l_\mathfrak{p}$, since

$$\sigma_p{}^{(k,l)} = \sum_{\phi(p)=\mathfrak{p}} u_p{}^{(k,l)}\sigma_p + \text{holomorphic part.}$$

Now we introduce the bilinear form $b(u, v)$ of $u, v \in \mathfrak{l}_\mathfrak{p}$ defined as follows:

$$b(u, v) = \sum_{\phi(p)=\mathfrak{p}} \operatorname{Res}_p[u_p v_p \sigma_p].$$

Clearly $b(u, v)$ is symmetric and non-singular, while we have, by virtue of Theorem 3. 2,

$$b(u^{(k,l)}, v^{(h,j)}) = \sum_{\phi(p)=\mathfrak{p}} \operatorname{Res}_p[(\phi_p{}^1)^{k+h}(\phi_p{}^2)^{l+j}\sigma_p] = 0,$$

proving that $b(u, v) = 0$ for all $u, v \in \mathfrak{L}_\mathfrak{p}$. Hence we get

$$l_\mathfrak{p} = \dim \mathfrak{L}_\mathfrak{p} \leqq \tfrac{1}{2} \dim \mathfrak{l}_\mathfrak{p} = \tfrac{1}{2} \sum_{\phi(p)=\mathfrak{p}} c_p,$$

proving (5. 9).

By a suitable choice of the system of local coordinates (w, z) with the origin $\mathfrak{p}$ (we write here w, z instead of z^1, z^2), we may assume that $R_\mathfrak{p}(w, z)$ is a distinguished polynomial:

$$R_\mathfrak{p}(w, z) = w^m + B_1(z)w^{m-1} + \cdots + B_m(z)$$

and that the parametric representation of each branch Γ_p, $p \in \phi^{-1}(\mathfrak{p})$, is given by

$$w = \mathfrak{P}_p(\tau_p) = a_{p1}\tau_p + a_{p2}\tau_p + \cdots, \qquad z = \tau_p{}^{m_p}.$$

Each coefficient $B_\nu(z)$ in $R_\mathfrak{p}(w, v)$ is a holomorphic function:

$$B_\nu(z) = b_{\nu 1}z + b_{\nu 2}z^2 + \cdots.$$

Replacing these $B_\nu(z)$ by the polynomials

$$B^*{}_\nu(z) = b_{\nu 1}z + b_{\nu 2}z^2 + \cdots + b_{\nu N}z^N,$$

we construct the polynomial

$$R^*(w,z) = w^m + B^*_1(z)w^{m-1} + \cdots + B^*_m(z).$$

Then, as one readily infers, we can find for each $p \varepsilon \phi^{-1}(\mathfrak{p})$ and for an arbitrary given integer $M > 0$ a power series $\mathfrak{P}^*_p(\tau_p)$ such that

$$R^*(\mathfrak{P}^*_p(\tau_p), \tau_p^{m_p}) = 0, \qquad V_p(\mathfrak{P}^*_p(\tau_p) - \mathfrak{P}_p(\tau_p)) \geqq M,$$

if N is sufficiently large compared with M. Suppose M to be chosen sufficiently large. Then the functions

$$\mathfrak{P}^*_p(\zeta_p^k z^{1/m_p}), \qquad k = 0, 1, \cdots, m_p - 1, \quad p \varepsilon \phi^{-1}(\mathfrak{p})$$

are all different to each other, where $\zeta_p = \exp(2\pi i/m_p)$, and we have

$$R^*(w,z) = \prod_{\mathfrak{p}} \prod_{k=0}^{m_p-1} [w - \mathfrak{B}^*_p(\zeta_p^k z^{1/m_p})].$$

Hence, for each $p \varepsilon \phi^{-1}(\mathfrak{p})$, the expression $w = \mathfrak{B}^*_p(\tau_p)$, $z = \tau_p^{m_p}$ represents an irreducible branch Γ^*_p and the curve Γ^* defined by the equation $R^*(w,z) = 0$ consists in $\mathfrak{U}_\mathfrak{p}$ of the irreducible branches Γ^*_p, $p \varepsilon \phi^{-1}(\mathfrak{p})$. Denote by $l^*_\mathfrak{p}$, c^*_p the integers corresponding to $l_\mathfrak{p}$, c_p associated with Γ^*, Γ^*_p. c^*_p is the order of the pole $\tau_p = 0$ of the differential

$$\sigma^*_p = [m_p \tau_p^{m_p-1}/\partial_1 R^*(\mathfrak{P}^*_p(\tau_p), \tau_p^{m_p})]d\tau_p$$

and $l^*_\mathfrak{p}$ is the number of formal sums $\sum_{\phi(p)=\mathfrak{p}} (\mathfrak{P}^*_p)^k \tau_p^{lm_p} \sigma^*_p$, $(k, l = 0, 1, 2, \cdots)$ which are independent modulo 0. Hence, if M and N are chosen sufficiently large, we have $c^*_p = c_p$, $l^*_\mathfrak{p} = l_\mathfrak{p}$. Supposing (w,z) as the inhomogeneous coordinates on the projective plane, we can consider Γ^* as a plane algebraic curve defined by the algebraic equation $R^*(w,z) = 0$. In order to prove (5.8), it is sufficient therefore to show

$$(5.8)^* \qquad l^*_\mathfrak{p} = \tfrac{1}{2} \sum_{\phi(p)=\mathfrak{p}} c^*_p$$

with respect to this plane algebraic curve Γ^*. Now we apply the formula (5.2) to Γ^*. Then we get

$$\dim\{\Gamma^*\} = I(\Gamma^*, \Gamma^*) - \pi_{\Gamma^*} + \sum_{\mathfrak{p}} [\tfrac{1}{2} \sum_{\phi(p)=\mathfrak{p}} c^*_p - l^*_\mathfrak{p}] + i + 2,$$

since, in this case, $q = j = 0$. On the other hand, denoting by n the degree of Γ^*, we have

$$\dim\{\Gamma^*\} = \tfrac{1}{2}(n^2 + 3n + 2), \quad \pi_{\Gamma^*} = \tfrac{1}{2}(n^2 - 3n + 2), \quad I(\Gamma^*, \Gamma^*) = n^2.$$

Hence we get $\sum_{\mathfrak{p}} [\frac{1}{2} \sum_{\phi(p)=\mathfrak{p}} c^*_p - l^*_\mathfrak{p}] + i = 0$. Combined with the inequality $i \geqq 0$ and (5.9), this yields immediately $(5.8)^*$, q. e. d.

The space $\Xi(\Gamma, -\mathfrak{d})$ was defined as the set of all differentials ξ on $\boldsymbol{\Gamma}$ satisfying

$$\xi \equiv \sum_{\mathfrak{p}} \sum_{k,l} a_{\mathfrak{p}kl} \sum_{\phi(p)=\mathfrak{p}} (\phi_p^{1})^k (\phi_p^{2})^l h_p \sigma_p \quad (\mathfrak{d}). \tag{5.10}$$

Clearly this condition implies

$$(\xi) - \mathfrak{d} + \mathfrak{c} \geqq 0. \tag{γ}$$

Now, let ξ be an arbitrary differential satisfying (γ). Then, for each singular point $\mathfrak{p}$ of Γ, the formal sum $\sum_{\phi(p)=\mathfrak{p}} h_p^{-1}\xi$ can be written as

$$\sum_{\phi(p)=\mathfrak{p}} h_p^{-1}\xi = \sum_{\phi(p)=\mathfrak{p}} u_p \sigma_p + \text{holomorphic part}, \tag{5.11}$$

where $u = \{u_p\} \varepsilon \mathfrak{l}_{\mathfrak{p}}$. Obviously this ξ satisfies (5.10) if and only if $u = \{u_p\}$ belongs to $\mathfrak{L}_{\mathfrak{p}}$ for each singular point $\mathfrak{p}$ of Γ. The identity (5.8) shows that $\dim \mathfrak{L}_{\mathfrak{p}} = \frac{1}{2} \dim \mathfrak{l}_{\mathfrak{p}}$; consequently, an arbitrary element $u \varepsilon \mathfrak{l}_{\mathfrak{p}}$ belongs to $\mathfrak{L}_{\mathfrak{p}}$ if and only if u satifies $b(u, v) = 0$ for all $v \varepsilon \mathfrak{L}_{\mathfrak{p}}$, while, for $u = \{u_p\}$ appearing in (5.11), we have

$$b(u, v) = \sum_{\phi(p)=\mathfrak{p}} \operatorname{Res}_p[v_p h_p^{-1}\xi].$$

Hence ξ satisfies (5.10) if and only if

$$\sum_{\phi(p)=\mathfrak{p}} \operatorname{Res}_p[u_p^{(k,l)} h_p^{-1}\xi] = 0, \qquad (k, l = 0, 1, 2, \cdots)$$

for each singular point $\mathfrak{p}$ of Γ. Thus we see that *the space* $\Xi(\Gamma, -\mathfrak{d})$ *consists of all differentials* ξ *on* $\boldsymbol{\Gamma}$ *satisfying*

$$(\xi) - \mathfrak{d} + \mathfrak{c} \geqq 0 \tag{γ}$$

and

$$\sum_{\phi(p)=\mathfrak{p}} \operatorname{Res}_p[(\phi_p^{1})^k (\phi_p^{2})^l h_p^{-1}\xi] = 0, \qquad (k, l = 0, 1, 2, \cdots) \tag{δ}$$

for each singular point $\mathfrak{p}$ *of* Γ.

Now, using the identity (5.8), we get from (5.2) immediately the following

THEOREM 5.1 (THE THEOREM OF RIEMANN-ROCH). *The dimension of the divisor class* $\{\Gamma\}$ *of a (reducible or irreducible) curve* Γ *is given by*

$$\dim\{\Gamma\} = I(\Gamma, \Gamma) - \pi_\Gamma - q + i + j + 2, \tag{5.12}$$

where, q *is the irregularity of* $\mathfrak{M}$, π_Γ *is the virtual genus of* Γ, i *is the dimension of the linear space* $\Xi(\Gamma, -\mathfrak{d})$ *and* j *is the number of linearly independent simple differentials* A *of the first kind on* $\mathfrak{M}$ *satisfying* $Q_f[\Omega \cdot A] = 0$ *for all* $f \varepsilon \mathfrak{f}(\Gamma, \mathfrak{d})$.

The number g of linearly independent double differentials of the first kind on $\mathfrak{M}$ is called the *geometric genus* of $\mathfrak{M}$. For an arbitrary divisor D on $\mathfrak{M}$, we denote by $\mathfrak{W}(D)$ the linear space consisting of all double differentials W such that $(W)+D \geqq 0$. Then we have

THEOREM 5.2 (RIEMANN-ROCH'S INEQUALITY). *We have*

$$\dim\{\Gamma\}+\dim\mathfrak{W}(-\Gamma) \geqq I(\Gamma,\Gamma)-\pi_\Gamma-q+g+2. \tag{5.13}$$

Proof. For every double differential $B=B_{12}(z)dz^1dz^2$ of the first kind, $\xi=B_{12}(\phi_p)h_p\sigma_p$ is a well defined differential on $\mathbf{\Gamma}$, as one readily infers by (3.17); furthermore, this differential satisfies clearly the above conditions (γ) and (δ). Thus $B\to\xi=B_{12}(\phi_p)h_p\sigma_p$ is a linear mapping which maps the space $\mathfrak{W}(0)$ of all double differentials of the first kind *into* the space $\Xi(\Gamma,-\mathfrak{d})$. Obviously $\xi=B_{12}(\phi_p)h_p\sigma_p$ vanishes identically on $\mathbf{\Gamma}$ if and only if $B\in\mathfrak{W}(-\Gamma)$. Consequently we get

$$i=\dim\Xi(\Gamma,-\mathfrak{d}) \geqq g-\dim\mathfrak{M}(-\Gamma).$$

Combined with (5.12), this yields immediately the inequality (5.13), q. e. d.

In what follows we assume that *there exists on* $\mathfrak{M}$ *at least one double differential* W_0 *not vanishing identically and put* $K=(W_0)$. Then, for an arbitrary fixed divisor D, every double differential $W\in\mathfrak{W}(D)$ corresponds one-to-one to a meromorphic function $F\in\mathfrak{F}(K+D)$ by means of the relation: $W=FW_0$; consequently we have

$$\dim\mathfrak{W}(D)=\dim\{K+D\}. \tag{5.14}$$

Using this identity, we get from Theorem 5.2 immediately the following

THEOREM 5.3 (RIEMANN-ROCH'S INEQUALITY). *We have*

$$\dim\{\Gamma\}+\dim\{K-\Gamma\} \geqq I(\Gamma,\Gamma)-\pi_\Gamma+g-q+2. \tag{5.15}$$

Now, we introduce the notion of the *deficiency* of an arbitrary divisor D on the (reducible or irreducible) curve Γ. Assume first that D and Γ have no component in common and denote by $\mathfrak{f}(\Gamma,D)$ the space consisting of all meromorphic functions f on $\mathbf{\Gamma}$ satisfying

$$(\alpha)_D \qquad f\equiv 0(-D\cdot\Gamma)$$

and

$$(\beta)_D \qquad \sum_{\phi(p)=\mathfrak{p}}\mathrm{Res}_p[(\phi_p{}^1)^k(\phi_p{}^2)^l f h_p(D)\sigma_p]=0, \qquad (k,l=0,1,2,\cdots)$$

for each singular point $\mathfrak{p}$ of Γ. Then every meromorphic function $F\in\mathfrak{F}(D)$

induces on Γ the meromorphic function $f(\tau) = F(\phi_p(\tau))$ belonging to $\mathfrak{f}(\Gamma, D)$ and the induced function $f(\tau)$ vanishes identically if and only if F belongs to $\mathfrak{F}(D - \Gamma)$. Hence we have

$$\dim \mathfrak{f}(\Gamma, D) \geqq \dim \{D\} - \dim \{D - \Gamma\}.$$

Now we define the *deficiency*[46] $\mathrm{def}(D/\Gamma)$ of the divisor D on the curve Γ as

$$\mathrm{def}(D/\Gamma) = \dim \mathfrak{f}(\Gamma, D) - [\dim \{D\} - \dim \{D - \Gamma\}].$$

As one readily infers, if D' *is linearly equivalent* to D, then $\mathrm{def}(D'/\Gamma) = \mathrm{def}(D/\Gamma)$. In case D and Γ have some components in common, we define $\mathrm{def}(D/\Gamma)$ as $\mathrm{def}(D/\Gamma) = \mathrm{def}(D'/\Gamma)$ by means of a divisor $D' \approx D$ having no component in common with Γ, assuming the existence of such a divisor D' (this is the case if $\mathfrak{M}$ is an algebraic surface). It is to be noted here that $\mathrm{def}(\Gamma/\Gamma)$ *coincides with the characteristic deficiency* $chd(\Gamma)$ *of* Γ, whenever $\mathrm{def}(\Gamma/\Gamma)$ can be defined. In fact, if there exists a divisor $D \approx \Gamma$ such that D and Γ have no component in common, then we have a meromorphic function F with the divisor $(F) = D - \Gamma$, and, putting

$$R_{\mathfrak{p}}(D) = R_{\mathfrak{p}}F, \tag{5.16}$$

we get a meromorphic function $R_{\mathfrak{p}}(D)$ representing D in $\mathfrak{U}_{\mathfrak{p}}$. It follows from (5.16) that the functions $h_p(D) = R_{\phi(p)}(D; \phi_p)$ satisfy

$$h_p(D)/h_q(D) = R_{\phi(p)}/R_{\phi(q)}, \quad \text{in } U_p \cap U_q,$$

proving that $\{h_p(D)\}$ can be used as a system $\{h_p\}$ defining the characteristic divisor $\mathfrak{d}$. Thus we see that $D \cdot \Gamma$ is the characteristic divisor $\mathfrak{d}$ so that $\mathfrak{f}(\Gamma, D)$ coincides with $\mathfrak{f}(\Gamma, \mathfrak{d})$. Hence we get

$$\mathrm{def}(D/\Gamma) = \dim \mathfrak{f}(\Gamma, \mathfrak{d}) - \dim \{\Gamma\} + 1,$$

proving that $\mathrm{def}(\Gamma/\Gamma) = chd(\Gamma)$.

The difference

$$\sup \{\Gamma\} = \dim \{\Gamma\} + \dim \{K - \Gamma\} - I(\Gamma, \Gamma) + \pi_\Gamma + q - g - 2$$

is called the *superabundance* of the divisor class $\{\Gamma\}$. From our deduction of Riemann-Roch's inequality it follows immediately that

$$\sup \{\Gamma\} = i - [g - \dim \{K - \Gamma\}] + q - chd(\Gamma).$$

Now, assuming the existence of a canonical divisor K such that K and Γ have

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[46] Cf. Goldman [9], § 6. In the terminology of Italian geometers, $\mathrm{def}(D/\Gamma)$ is the deficiency of the linear series on Γ cut out by the complete linear system $|D|$.

no component in common, we shall show that $i - [g - \dim\{K - \Gamma\}]$ is equal to $\mathrm{def}(K/\Gamma)$. Let $W = W_{12}dz^1dz^2$ be the double differential with $(W) = K$ and consider the one-to-one correspondence $f \to \xi = fW_{12}(\phi_p)h_p\sigma_p$ between meromorphic functions f and differentials ξ on $\mathbf{\Gamma}$. Since the divisor $(W_{12}(\phi_p)h_p\sigma_p)$ of $W_{12}h_p\sigma_p$ is $K \cdot \Gamma + \mathfrak{d} - \mathfrak{c}$, ξ satisfies $(\xi) - \mathfrak{d} + \mathfrak{c} \geqq 0$ if and only if $(f) + K \cdot \Gamma \geqq 0$. Again, since $W_{12}(\phi_p) = h_p(K)$, ξ satisfies the condition

$$\sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[(\phi_p{}^1)^k(\phi_p{}^2)^l h_p{}^{-1}\xi] = 0$$

if and only if f satisfies

$$\sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[(\phi_p{}^1)^k(\phi_p{}^2)^l f h_p(K)\sigma_p] = 0.$$

Thus ξ belongs to $\Xi(\Gamma, -\mathfrak{d})$ if and only if $f \in \mathfrak{f}(\Gamma, K)$. Hence we have $i = \dim \Xi(\Gamma, -\mathfrak{d}) = \dim \mathfrak{f}(\Gamma, K)$, while, as (5.14) shows, $g = \dim \mathfrak{W}(0) = \dim\{K\}$. We get therefore

$$i - [g - \dim\{K - \Gamma\}] = \dim \mathfrak{f}(\Gamma, K) - [\dim\{K\} - \dim\{K - \Gamma\}],$$

proving that $i - [g - \dim\{K - \Gamma\}]$ is equal to $\mathrm{def}(K/\Gamma)$. Thus we obtain the following

THEOREM 5.4 (G. Castelnuovo [47]). *The superabundance of the divisor class of an arbitrary curve* Γ *is represented as*

$$\sup\{\Gamma\} = \mathrm{def}(K/\Gamma) + q - chd(\Gamma). \tag{5.17}$$

6. Double differentials. In this Section we shall determine the dimension of the linear space $\mathfrak{W}(\Gamma)$ consisting of all double differentials on $\mathfrak{M}$ which are multiples of $-\Gamma$ in the sense that $(W) + \Gamma \geqq 0$. As was shown in Section **3**, the residue $\xi = \mathfrak{R}(W) = R_pW_{12}(\phi_p)\sigma_p$ of an arbitrary double differential $W \in \mathfrak{W}(\Gamma)$ is a well defined differential on $\mathbf{\Gamma}$. This differential $\xi = \mathfrak{R}(W)$ satisfies obviously

$$(\xi) + \mathfrak{c} \geqq 0; \tag{γ}_0$$

again, as one readily infers by Theorem 3.2, $\xi = \mathfrak{R}(W)$ satisfies

$$\sum_{\phi(p)=\mathfrak{p}} \mathrm{Res}_p[(\phi_p{}^1)^k(\phi_p{}^2)^l\xi] = 0, \qquad (k, l = 0, 1, 2, \cdots) \tag{δ}_0$$

for each singular $\mathfrak{p}$ of Γ. *We denote by* $\Xi(\Gamma)$ *the space consisting of all differentials* ξ *on* $\mathbf{\Gamma}$ *satisfying* $(\gamma)_0$ *and* $(\delta)_0$. The mapping $W \to \xi = \mathfrak{R}(W)$

[47] This theorem was proved by G. Castelnuovo in the case that $\mathfrak{M}$ is an algebraic surface. Cf. Castelnuovo [4]; see also Zariski [30], p. 71.

is a linear transformation which maps $\mathfrak{M}(\Gamma)$ *into* $\Xi(\Gamma)$. Clearly $\xi = \Re(W)$ vanishes identically if and only if W is a differential of the first kind.

Now, suppose an arbitrary differential $\xi \in \Xi(\Gamma)$ as given. Then, does there exist a double differential $W \in \mathfrak{W}(\Gamma)$ such that $\Re(W) = \xi$? In order to solve this question, we associate with ξ the 3-current T_ξ defined as

$$T_\xi[X] = \int_\Gamma \xi \cdot \sum_a X_{a^*} d\bar{\phi}_p{}^a, \tag{6.1}$$

where X is a "variable" 1-form of class C^∞ and the value of the integral in the right-hand side is to be interpreted as the Cauchy principal value, in case it does not converge absolutely (see (3.15)). T_ξ is obviously a pure 3-current of type 1. Again we have

$$dT_\xi = 0. \tag{6.2}$$

In fact, for an arbitrary function $\Psi(z)$ of class C^∞, we have

$$dT_\xi[\Psi] = T_\xi[d\Psi] = \int_\Gamma \xi \cdot d\Psi(\phi_p) = -\int_\Gamma \mathrm{d}(\Psi(\phi_p)\xi),$$

while $\Psi(\phi_p)\xi$ is a 1-form of class C^∞ in $\mathbf{\Gamma} - |\,\mathfrak{c}\,|$. Hence we get

$$dT_\xi[\Psi] = \sum_{\mathfrak{p}} \sum_{\phi(p)=\mathfrak{p}} \lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} \Psi(\phi_p)\xi,$$

where $\sum_{\mathfrak{p}}$ denotes the sum extended over all singular points $\mathfrak{p}$ of Γ; while, using Lemma 3.1, we get from $(\delta)_0$ the identity

$$\sum_{\phi(p)=\mathfrak{p}} \lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} \Psi(\phi_p)\xi = 0.$$

Hence, we obtain $dT_\xi[\Psi] = 0$, proving (6.2). Now, if there exists $W \in \mathfrak{W}(\Gamma)$ such that $\Re(W) = \xi$, then as (3.15) shows, we have

$$dW = 2\pi i T_\xi \tag{6.3}$$

and therefore

$$HT_\xi = 0. \tag{6.4}$$

Now, assuming conversely that T_ξ satisfies (6.4), we shall prove the existence of $W \in \mathfrak{W}(\Gamma)$ with $\Re(W) = \xi$. For this purpose, we introduce the 2-current θ_ξ defined as $\theta_\xi = (d\Lambda G + i\delta G)T_\xi$. By virtue of Theorem 1.4, θ_ξ is a holomorphic 2-form in $\mathfrak{M} - \Gamma$. In order to show that θ_ξ is meromorphic on Γ, we first consider an arbitrary simple point $\mathfrak{p}$ of Γ. Choose the system of local coordinates (w, z) in a neighborhood $\mathfrak{U}_\mathfrak{p}$ of $\mathfrak{p}$ so that $R_\mathfrak{p} = w$ and that $z = \tau_p$ on Γ and put $\xi = \xi_1(\tau_p)d\tau_p$ in U_p, where $p = \phi^{-1}(\mathfrak{p})$.

By virtue of $(\gamma)_0$, $\xi_1(\tau_p)$ is a holomorphic function in U_p. Hence, putting $W_\mathfrak{p}(w,z) = (dw/w)\cdot\xi_1(z)dz$, we get a double differential $W_\mathfrak{p}(w,z)$ defined in $\mathfrak{U}_\mathfrak{p}$. Considering $W_\mathfrak{p}$ as a 2-current in $\mathfrak{U}_\mathfrak{p}$, we get by (3. 10), $dW_\mathfrak{p} = 2\pi i T_\xi$, in $\mathfrak{U}_\mathfrak{p}$, since, in this case, $\sigma_p = d\tau_p$. On the other hand, θ_ξ is a pure 2-current of type 0 and satisfies

$$d\theta_\xi = iT_\xi, \tag{6. 5}$$

as Theorem 1. 4 shows. Hence we get $d(2\pi\theta_\xi - W_\mathfrak{p}) = 0$, in $\mathfrak{U}_\mathfrak{p}$. By virtue of Theorem 1. 3, we infer therefore that $B_\mathfrak{p} = 2\pi\theta_\xi - W_\mathfrak{p}$ is a holomorphic 2-form in $\mathfrak{U}_\mathfrak{p}$. Consequently $2\pi\theta_\xi$ is a meromorphic 2-form in $\mathfrak{U}_\mathfrak{p}$; moreover, since $\sigma_p = d\tau_p = dz$ and $2\pi R_\mathfrak{p}\theta_\xi = \xi_1(z)dwdz + R_\mathfrak{p}B_\mathfrak{p}$, we get

$$2\pi R_\mathfrak{p}\theta_{\xi 12}(\phi_p)\sigma_p = \xi, \qquad (p = \phi^{-1}(\mathfrak{p})). \tag{6. 6}$$

Now, assume $\mathfrak{p}$ to be a singular point of Γ. Then it follows from the above result that $R_\mathfrak{p}\theta_\xi$ is holomorphic in $\mathfrak{U}_\mathfrak{p} - \mathfrak{p}$. Hence, by virtue of a theorem of Hartogs,[48] $R_\mathfrak{p}\theta_\xi$ must be holomorphic in the whole of $\mathfrak{U}_\mathfrak{p}$. Thus $R_\mathfrak{p}\theta_\xi$ is holomorphic in $\mathfrak{U}_\mathfrak{p}$ for every $\mathfrak{p}\in\mathfrak{M}$, proving that θ_ξ is a double differential belonging to $\mathfrak{W}(\Gamma)$. Moreover it follows from (6. 6) that $2\pi\theta_\xi$ satisfies $\Re(2\pi\theta_\xi) = \xi$. Thus the existence of $W\in\mathfrak{W}(\Gamma)$ satisfying $\Re(W) = \xi$ is proved.

For an arbitrary simple differential $A = \sum A_\alpha(z)dz^\alpha$ of the first kind, we put

$$A_\Gamma = \sum_\alpha A_\alpha(\phi_p)d\phi_p{}^\alpha.$$

A_Γ is the *differential induced on* Γ *by* A. It is clear that HT_ξ vanishes if and only if $T_\xi[\Phi] = 0$ for all harmonic 1-form Φ of the first kind, while every harmonic 1-form Φ of the first kind can be written as $\Phi = A' + \bar{A}$ by means of two simple differentials A', A of the first kind. Hence HT_ξ vanishes if and only if

$$T_\xi[\bar{A}] = \int_\Gamma \xi\cdot\bar{A}_\Gamma = 0$$

for all simple differentials A of the first kind. Thus we obtain the following

THEOREM 6. 1. *The linear mapping* $W\to\xi = \Re(W)$ *maps the space* $\mathfrak{W}(\Gamma)$ *onto the subspace of* $\Xi(\Gamma)$ *consisting of all differentials* $\xi\in\Xi(\Gamma)$ *satisfying*

$$\int_\Gamma \xi\cdot\bar{A}_\Gamma = 0 \tag{6. 7}$$

[48] Hartogs [11].

for all simple differentials A of the first kind on $\mathfrak{M}$. Every $W \in \mathfrak{W}(\Gamma)$ can be represented by means of $\xi = \mathfrak{R}(W)$ as

$$W = 2\pi\theta_\xi + B, \tag{6.8}$$

where B is a double differential of the first kind. Conversely, for every $\xi \in \Xi(\Gamma)$ satisfying (6.7), the double differential W given by (6.8) belongs to $\mathfrak{W}(\Gamma)$ and satisfies $\mathfrak{R}(W) = \xi$.

Denote by k the number of linearly independent simple differentials A of the first kind such that A_Γ vanishes identically. Then the number of linearly independent *induced* differentials A_Γ on $\mathbf{\Gamma}$ is equal to $q - k$. Moreover each A_Γ is a differential of the first kind and therefore is contained in $\Xi(\Gamma)$. Hence the number of linearly independent differentials $\xi \in \Xi(\Gamma)$ satisfying (6.7) is equal to $\dim \Xi(\Gamma) - q + k$. By virtue of Theorem 6.1, we infer therefore that the dimension of the space $\mathfrak{W}(\Gamma)$ is given by

$$\dim \mathfrak{W}(\Gamma) = g + \dim \Xi(\Gamma) - q + k. \tag{6.9}$$

The dimension of $\Xi(\Gamma)$ is closely connected with the virtual genus π_Γ of Γ. Consider an arbitrary system $\{\xi_p | p \in |\mathfrak{c}|\}$ of differentials ξ_p defined respectively at $p \in |\mathfrak{c}|$ and denote by $\sum_p \xi_p$ the formal sum of ξ_p introduced in Section **4**. By virtue of (5.8), the number of $\sum_p \xi_p$ satisfying $(\gamma)_0$ and $(\delta)_0$ which are independent modulo 0 is given by $\frac{1}{2}\sum_p c_p$, while, for arbitrary $\sum_p \xi_p$, there exists a differential ξ on $\mathbf{\Gamma} = \sum_{\nu=1}^{\mu} \mathbf{\Gamma}_\nu$ satisfying $\xi \equiv \sum_p \xi_p$ (0) if and only if

$$\sum_p^{(\nu)} \operatorname{Res}_p[\xi_p] = 0, \qquad (\nu = 1, 2, \cdots, \mu), \tag{6.10}$$

where $\sum_p^{(\nu)}$ denotes the sum extended over all $p \in |\mathfrak{c}| \cap \mathbf{\Gamma}_\nu$. On the other hand, for arbitrarily preassigned values ρ_p, $p \in |\mathfrak{c}|$, satisfying $\sum_{\phi(p)=\mathfrak{p}} \rho_p = 0$ for each singular point $\mathfrak{p}$ of Γ, we can choose $\sum_p \xi_p$ satisfying $(\gamma)_0$ and $(\delta)_0$ so that $\operatorname{Res}_p[\xi_p] = \rho_p$ for each $p \in |\mathfrak{c}|$. Decompose Γ into the sum $\Gamma = \sum_{s=1}^{m} \Gamma^{(s)}$ of *connected components* $\Gamma^{(s)} = \sum_{\nu=\mu(s-1)+1}^{\mu(s)} \Gamma_\nu$, where $\mu(0) = 0$ and $\mu(m) = \mu$. Then it fofllows from the above results that one can choose $\sum_p \xi_p$ satisfying

$(\gamma)_0$ and $(\delta)_0$ so that each $\sum_p^{(\nu)} \mathrm{Res}_p[\xi_p]$ $(\mu(s-1) < \nu < \mu(s),\ s = 1, 2, \cdots, m)$ takes an arbitrarily preassigned value, while $(\gamma)_0$ and $(\delta)_0$ imply

$$\sum_p^{(\mu(s))} \mathrm{Res}_p[\xi_p] = -\sum_{\mu(s-1)<\nu<\mu(s)} \sum_p^{(\nu)} \mathrm{Res}_p[\xi_p].$$

Hence the number of linearly independent conditions involved in (6.10) is $\mu - m$, while the number of linearly independent differentials satisfying $\xi \equiv 0(0)$ is $\sum_{\nu=1}^{\mu} \pi_\nu$. Consequently we get

$$\dim \Xi(\Gamma) = \sum_{\nu=1}^{\mu} \pi_\nu + \tfrac{1}{2} \sum c_p - \mu + m = \pi_\Gamma + m - 1.$$

Inserting this into (6.9), we obtain the following

THEOREM 6.2. *The dimension of the space* $\mathfrak{W}(\Gamma)$ *is given by*

$$\dim \mathfrak{W}(\Gamma) = g + \pi_\Gamma + m - 1 - q + k, \tag{6.11}$$

where π_Γ *is the virtual genus of* Γ, m *is the number of connected component of* Γ *and* k *is the number of linearly independent simple differentials* A *of the first kind such that* A_Γ *vanishes identically on* $\boldsymbol{\Gamma}$.

It is obvious that A_Γ vanishes identically if and only if the simple Picard integral $\int^z A$ is constant on each connected component of Γ. Hence k *is equal to the number of linearly independent simple Picard integrals of the first kind which are constant on each connected component of* Γ. The difference $dd(\Gamma) = g + \dim \Xi(\Gamma) - \dim \mathfrak{W}(\Gamma)$ is called the differential deficiency[49] of Γ. The formula (6.9) shows that

$$dd(\Gamma) = q - k; \tag{6.12}$$

thus $dd(\Gamma)$ *is always not greater than* q. Let Δ be an arbitrary curve which is *homologous* to Γ. *Then* A_Δ *is identically zero if and only if* A_Γ *is identically zero,* A being an arbitrary simple differential of the first kind. In fact, denoting by $\tau = u + iv$ the local uniformization variable on $\boldsymbol{\Gamma}$, we have

$$\int_\Gamma A \cdot \bar{A} = \int_{\boldsymbol{\Gamma}} A_\Gamma \cdot \bar{A}_\Gamma = -2i \int_{\boldsymbol{\Gamma}} |\, A_\Gamma / d\tau \,|^2 du dv,$$

[49] Goldman [9], § 6. In the terminology of Italian geometers, $dd(\Gamma)$ is the deficiency of the linear series on Γ cut out by the adjoint system of the complete linear system $|\,\Gamma\,|$. See the formula (6.17) below.

proving that A vanishes identically if and only if $\int_\Gamma A \cdot \bar{A}$ is zero, while, since $d[A \cdot \bar{A}] = 0$, we have

$$\int_\Delta A \cdot \bar{A} = \int_\Gamma A \cdot \bar{A}.$$

Hence A_Δ is zero if and only if A_Γ is zero. This shows that the number k of indepenedent simple differentials A of the first kind satisfying $A_\Gamma = 0$ is determined uniquely by the homology class of Γ so that the *differential deficiency* $dd(\Gamma)$ *of* Γ *is uniquely determined by the homology class of* Γ. Now, it follows from (6. 11) that

$$m = \dim \mathfrak{W}(\Gamma) - \pi_\Gamma - g + q + 1 - k, \ .$$

proving that the *number* m *of the connected component of* Γ *is determined uniquely by the divisor class* $\{\Gamma\}$ *of* Γ.[50]

THEOREM 6. 3. *If* Γ *is connected and* $\dim \{\Gamma\} \geqq 2$, *then the differential deficiency* $dd(\Gamma)$ *of* Γ *is equal to the irregularity* q *of* $\mathfrak{M}$: $dd(\Gamma) = q$.

Proof. By virtue of (6. 12), it is sufficient to show that any simple Picard integral $\int^z A$ of the first kind is not a constant on Γ unless A vanishes identically. By hypothesis, there exists on $\mathfrak{M}$ at least one non-constant meromorphic function F such that $(F) + \Gamma \geqq 0$. Putting $(F - t) = \Gamma_t - \Gamma$, t being a complex parameter, we get a linear pencil $\{\Gamma_t\}$. Γ_t is a curve (i. e. a positive divisor without multiple component) unless t takes one of a finite number of special values $t_1, t_2, \cdots, t_s$; moreover, by the above result, Γ_t is connected for $t \neq t_\nu$. Now, if $\int^z A$ is constant on Γ, then $\int^z A$ is constant on each Γ_t, $t \neq t_\nu$, since Γ_t is homologous to Γ. Let $\alpha(t)$ be the value of $\int^z A$ on Γ_t, $t \neq t_\nu$. Then $\alpha(t)$ is an additive holomorphic function of t for $t \neq t_\nu$ $(\nu = 1, 2, \cdots, s)$ ($\alpha(t)$ is holomorphic also for $t = \infty$). In the neighborhood of t_ν, we have therefore

$$\alpha(t) = c \log(t - t_\nu) + \sum_{m=-\infty}^{+\infty} c_m (t - t_\nu)^m.$$

On the other hand, since $\alpha(t) = \int^z A$, $z \in \Gamma_t$, it is clear that the limit

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[50] Especially if Γ is linearly equivalent to an irreducible curve, then Γ is connected. Cf. Zariski [30], p. 36.

$\lim_{\epsilon \to +0} \alpha(\epsilon e^{i\theta} + t_\nu)$ exists and is independent of θ. Hence, in the above expansion of $\alpha(t)$, c and c_m, $m < 0$, must be zero. Thus $\alpha(t)$ is everywhere holomorphic and therefore $\alpha(t)$ must be a constant. This shows that $\int^z A$ reduces to a constant, proving that A vanishes identically on $\mathfrak{M}$, q. e. d.

Combined with (6. 12), the above Theorem 6. 3 yields immediately a theorem of F. Enriques [51] to the effect that the *differential deficiency* $dd(\Gamma)$ *of an arbitrary irreducible curve* Γ *on an algebraic surface* $\mathfrak{M}$ *is never greater than the irregularity* q *of* $\mathfrak{M}$ *and there exists on* $\mathfrak{M}$ *an irreducible curve* Γ *for which* $dd(\Gamma) = q$..

In what follows we assume that *there exists on* $\mathfrak{M}$ *at least one canonical divisor* K. Then, using (5. 14), we get from Theorem 6. 2 the following

THEOREM 6. 4. *The dimension of the adjoint class* $\{K + \Gamma\}$ *of* $\{\Gamma\}$ *is given by*

$$\dim \{K + \Gamma\} = g + \pi_\Gamma + m - 1 - q + k. \tag{6. 13}$$

This Theorem 6. 4 yields immediately *Riemann-Roch's inequality for the adjoint class* $\{\Gamma'\} = \{K + \Gamma\}$ of $\{\Gamma\}$. Defining the *virtual genus* π' of Γ' by

$$2\pi' - 2 = I(\Gamma', \Gamma') + I(K, \Gamma'),$$

we get readily from (6. 13) the formula

$$\dim \{\Gamma'\} = I(\Gamma', \Gamma') - \pi' + g - q + 2 + m - 1 + k,$$

which yields immediately *Riemann-Roch's inequality*

$$\dim \{\Gamma'\} \geqq I(\Gamma', \Gamma') - \pi' + g - q + 2.$$

Furthermore we get the following

THEOREM 6. 5. *The superabundance of the adjoint class* $\{K + \Gamma\}$ *of the divisor class* $\{\Gamma\}$ *of an arbitrary curve* Γ *is given by*

$$\sup \{K + \Gamma\} = m - 1 + k, \tag{6. 14}$$

where m *is the number of conected components of* Γ *and* k *is the number of linearly independent simple Picard integrals* $\int^z A$ *of the first kind which are constant on each connected component of* Γ.

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[51] Enriques [8]; Goldman [9], § 6. Cf. also Zariski [30], p. 66.

Combining (6.14) with (6.12), we get immediately

$$\sup\{K+\Gamma\} = m-1+q-dd(\Gamma). \tag{6.15}$$

Especially if Γ is a *connected* curve, we have

$$\sup\{K+\Gamma\} = q-dd(\Gamma). \tag{6.16}$$

Hence we get from Theorem 6.3 the following

THEOREM 6.6. *If Γ is connected and* $\dim\{\Gamma\} \geqq 2$, *then the superabundance of the adjoint class $\{K+\Gamma\}$ of Γ is zero*: $\sup\{K+\Gamma\} = 0$.

It is to be noted here that

$$dd(\Gamma) = \mathrm{def}(K+\Gamma/\Gamma) \tag{6.17}$$

whenever $\mathrm{def}(K+\Gamma/\Gamma)$ can be defined. In fact, if there exist a canonical divisor $K=(W)$ and a divisor $D \varepsilon \{\Gamma\}'$ such that K and D have no component in common with Γ, then it can be easily verified in the same manner as in the proof of Theorem 5.4 that the mapping $f \to \xi = fW_{12}(\phi_p)h_p(D)\sigma_p$ maps $\mathfrak{f}(\Gamma, K+D)$ *onto* $\Xi(\Gamma)$ one-to-one so that $\dim \Xi(\Gamma) = \dim \mathfrak{f}(\Gamma, K+D)$. Hence we get

$$dd(\Gamma) = \dim \mathfrak{f}(\Gamma, K+D) - \dim\{K+D\} + \dim\{K\},$$

proving (6.17).

In case $\mathfrak{M}$ is an algebraic surface, the above Theorem 6.6 is reduced to a special case of a theorem of F. Severi [52] to the effect that, *if an irreducible curve Γ on $\mathfrak{M}$ is a member of a continuous system which is not an irrational pencil, then the superabundance of the adjoint system $\{K+\Gamma\}$ of $\{\Gamma\}$ is zero.* This theorem of Severi can be readily deduced from Theorem 6.5. Indeed, as was proved by Severi,[53] if Γ satisfies the hypothesis of the theorem, then there exists on $\mathfrak{M}$ no non-trivial simple Picard integral of the first kind which reduces to a constant on Γ; hence, by Theorem 6.5, $\sup\{K+\Gamma\} = 0$.

7. Algebraic surfaces. In this Section $\mathfrak{M}$ is assumed to be an *algebraic surface* without singularities in the projective space $\mathfrak{E}^N$ of complex dimension N. We denote the projective coordinates in $\mathfrak{E}^N$ by $\zeta_0, \zeta_1, \cdots, \zeta_N$. Then the Kählerian metric $ds^2 = 2\sum g_{\alpha\beta^*}(dz^\alpha d\bar{z}^\beta)$ on $\mathfrak{M}$ is given by

$$g_{\alpha\beta^*} = \partial^2 Z/\partial z^\alpha \partial \bar{z}^\beta, \qquad Z = (1/2\pi)\log\{\sum_{k=0}^{N} |\zeta_k/\zeta|^2\},$$

[52] Severi [23]. See also Zariski [30], p. 144.
[53] Severi [22]; Zariski [30], p. 144.

where $\zeta = l_0\zeta_0 + l_1\zeta_1 + \cdots + l_N\zeta_N$ is an arbitrary *fixed* linear form of $\zeta_0, \zeta_1, \cdots, \zeta_N$ (it can be readily verified that $g_{\alpha\beta^*}$ are independent of the choice of ζ). The hyperplane section of $\mathfrak{M}$ cut out by the hyperplane $\zeta = 0$ will be denoted by E. We choose the linear form ζ so that E and Γ have only *simple intersection points*. Now, let us consider the quantity $Q_f[\Omega \cdot A]$ appearing in the condition (3.32). Putting

$$\kappa = \sum \kappa_\alpha \cdot dz^\alpha, \qquad \kappa_\alpha = \partial Z/\partial z^\alpha,$$

we get $\Omega = -id\kappa$ and therefore

$$i\Omega \cdot A = d(\kappa \cdot A) = \sum_\alpha \partial(\kappa_1 A_2 - \kappa_2 A_1)/\partial\bar{z}^\alpha \cdot dz^1 dz^2 d\bar{z}^\alpha.$$

Hence, considering $(\kappa_1 A_2 - \kappa_2 A_1) f h_p \sigma_p$ as a 1-form on $\mathbf{\Gamma}$, we have

$$\begin{aligned} iQ_f[\Omega \cdot A] &= \int_{\mathbf{\Gamma}} \sum_\alpha \partial(\kappa_1 A_2 - \kappa_2 A_1)/\partial\bar{z}^\alpha \cdot f h_p \sigma_p d\bar{\phi}_p{}^\alpha \\ &= -\int_{\mathbf{\Gamma}} d\{(\kappa_1 A_2 - \kappa_2 A_1) f h_p \sigma_p\}. \end{aligned}$$

The 1-form $(\kappa_1 A_2 - \kappa_2 A_1) f h_p \sigma_p$ is of class C^∞ on $\mathbf{\Gamma}$ except for the points p on $|\mathfrak{c}|$ and on $|E \cdot \Gamma|$, while, since, by our assumption, E and Γ have only simple intersection points, $|\mathfrak{c}|$ and $|E \cdot \Gamma|$ have no common point. We have therefore

$$iQ_f[\Omega \cdot A] = \Big(\sum_{p \in |\mathfrak{c}|} + \sum_{p \in |E \cdot \Gamma|}\Big) \lim_{\epsilon \to +0} \int_{|\tau_p| = \epsilon} (\kappa_1 A_2 - \kappa_2 A_1) f h_p \sigma_p,$$

while, using Lemma 3.1, one can readily deduce from the condition (β) for f the identity

$$\sum_{\phi(p) = \mathfrak{p}} \lim_{\epsilon \to +0} \int_{|\tau_p| = \epsilon} (\kappa_1 A_2 - \kappa_2 A_1) f h_p \sigma_p = 0$$

for each singular point $\mathfrak{p}$ of Γ. Hence we get

$$iQ_f[\Omega \cdot A] = \sum_{p \in |E \cdot \Gamma|} \lim_{\epsilon \to +0} \int_{|\tau_p| = \epsilon} (\kappa_1 A_2 - \kappa_2 A_1) f h_p \sigma_p. \tag{7.1}$$

Consider an arbitrary point $p \in |E \cdot \Gamma|$. Assuming that $\mathfrak{p} = \phi(p)$ is not on the hyperplane $\zeta_0 = 0$, we put $\zeta_0 = 1$ in $\mathfrak{U}_\mathfrak{p}$. Then the inhomogeneous coordinates $\zeta_1, \zeta_2, \cdots, \zeta_N$ of the point $z = (z^1, z^2)$ in $\mathfrak{U}_\mathfrak{p}$ are holomorphic functions of z^1, z^2 and therefore

$$\kappa = -(1/2\pi) d \log \zeta + \text{continuous form}, \qquad \text{in } \mathfrak{U}_\mathfrak{p}.$$

Choosing the local coordinates z^1, z^2 so that $R_p = z^1$, $\zeta = z^2$ in $\mathfrak{U}_\mathfrak{p}$ (this is

possible since $\mathfrak{p}$ is assumed to be a simple intersection point of E and Γ), we get therefore

$$\lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} (\kappa_1 A_2 - \kappa_2 A_1) f h_p \sigma_p = i A_1(\mathfrak{p}) f h_p(p),$$

where $fh_p(p)$ denotes the value of the holomorphic function fh_p at p. A_1 can be interpreted as the coefficient of the differential $A_E = A_1 dz^1$ induced on E by A and represented as

$$A_1(\mathfrak{p}) = \mathrm{Res}_{\mathfrak{p}}[A_E/R_{\mathfrak{p}}] = (1/2\pi i) \oint_{\mathfrak{p}} R_{\mathfrak{p}}^{-1} A_E,$$

where $\oint_{\mathfrak{p}}$ denotes the integral extended over the small circle *on* E with the center $\mathfrak{p}$. Thus we get

$$\lim_{\epsilon \to +0} \int_{|\tau_p|=\epsilon} (\kappa_1 A_2 - \kappa_2 A_1) f h_p \sigma_p = i f h_p(p) \mathrm{Res}_{\mathfrak{p}}[A_E/R_p].$$

Combined with (7. 1), this yields the *important formula*

$$Q_f[\Omega \cdot A] = \sum_{p \in |E \cdot \Gamma|} f h_p(p) \mathrm{Res}_{\phi(p)}[A_E/R_p], \tag{7.2}$$

where the expression $fh_p(p)\mathrm{Res}_{\phi(p)}[A_E/R_p]$ *is independent of the choice of the local equation* $R_p = 0$.

Now, assume that Γ is *an irreducible curve without singular points* and consider an arbitrary divisor D not containing Γ as one of its components. In this case we can consider that $\mathbf{\Gamma} = \Gamma$, so that $\mathfrak{f}(\Gamma, D)$ is simply the space consisting of all meromorphic functions f on Γ which are multiples of $-D \cdot \Gamma$. The deficiency [54] of D *on* Γ is given therefore by

$$\mathrm{def}(D/\Gamma) = \dim_\Gamma \{D \cdot \Gamma\} - \dim \{D\} + \dim \{D - \Gamma\},$$

where $\dim_\Gamma \{D \cdot \Gamma\}$ denotes the dimension of the divisor class $\{D \cdot \Gamma\}$ on the curve Γ. Now, assuming that $\Gamma - D$ is linearly equivalent to a (reducible or irreducible) curve Δ which does not contain Γ as one of its components and *is connected in* $\mathfrak{M} - \Gamma$, we shall determine $\mathrm{def}(D/\Gamma)$. Let F_0 be a meromorphic function on $\mathfrak{M}$ such that $(F_0) = D + \Delta - \Gamma$. We choose the meromorphic functions $R_{\mathfrak{p}}(D; z)$, $R_{\mathfrak{p}}(\Delta; z)$ expressing respectively the divisors D, Δ in $\mathfrak{U}_{\mathfrak{p}}$ so that $F_0 = R_{\mathfrak{p}}(D) R_{\mathfrak{p}}(\Delta)/R_{\mathfrak{p}}$. Then, as was shown in Section 5, one gets a system $\{h_{\mathfrak{p}}\}$ defining the characteristic divisor $\mathfrak{d}$ by putting $h_{\mathfrak{p}} = R_{\mathfrak{p}}(D; \phi_{\mathfrak{p}}) R_{\mathfrak{p}}(\Delta; \phi_{\mathfrak{p}})$ (since $\mathbf{\Gamma} = \Gamma$, it is not necessary to distinguish p from $\mathfrak{p} = \phi(p)$). Denote by M an arbitrary meromorphic function in

[54] Goldman [9], § 6.

$\mathfrak{F}(D+\Delta)$. Then the mapping $M \to F = MF_0$ maps $\mathfrak{F}(D+\Delta)$ *onto* $\mathfrak{F}(\Gamma)$ one-to-one. Again, by virtue of Theorem 3.4, every $F \in \mathfrak{F}(\Gamma)$ corresponds one-to-one up to an additive constant to $f \in \mathfrak{f}(\Gamma, \mathfrak{d})$ satisfying $Q_f[\Omega \cdot A_\lambda] = 0$ $(\lambda = 1, 2, \cdots, q)$ by means of the relation $f = R_\mathfrak{p} F / h_\mathfrak{p}$, while, since $h_\mathfrak{p}/R_\mathfrak{p} = R_\mathfrak{p}(D)R_\mathfrak{p}(\Delta)/R_\mathfrak{p} = F_0$, the relation $f = R_\mathfrak{p}F/h_\mathfrak{p}$ is equivalent to $f(\tau_\mathfrak{p}) = M(\phi_\mathfrak{p}(\tau_\mathfrak{p}))$. Hence, every meromorphic function $M \in \mathfrak{F}(D+\Delta)$ corresponds one-to-one up to an additive term cF_0^{-1} (c being an arbitrary constant) to a meromorphic function $f \in \mathfrak{f}(\Gamma, \mathfrak{d})$ satisfying $Q_f[\Omega \cdot A_\lambda] = 0$ $(\lambda = 1, 2, \cdots, q)$ by means of the relation $f = M(\phi_\mathfrak{p})$. Obviously $\mathfrak{d} = D \cdot \Gamma + \Delta \cdot \Gamma$ and therefore $\mathfrak{f}(\Gamma, \mathfrak{d}) = \mathfrak{f}(\Gamma, D+\Delta)$. Now, if $M \in \mathfrak{F}(D)$, then $f = M(\phi_\mathfrak{p})$ belongs clearly to the subspace $\mathfrak{f}(\Gamma, D)$ of $\mathfrak{f}(\Gamma, D+\Delta)$; moreover any $M \in \mathfrak{F}(D)$ is determined uniquely by $f = M(\phi_\mathfrak{p})$, since F_0^{-1} does not belong to $\mathfrak{F}(D)$. Conversely, *if* $f = M(\phi_\mathfrak{p})$ *belongs to* $\mathfrak{f}(\Gamma, D)$, *then we can find* $M_1 \in \mathfrak{F}(D)$ *such that* $M_1(\phi_\mathfrak{p}) = f$. To prove this, we shall first show that, if $f = R_\mathfrak{p}F/h_\mathfrak{p}$ belongs to $\mathfrak{f}(\Gamma, D)$, then the function F_Δ induced on Δ by F is holomorphic everywhere on Δ. Clearly F_Δ is holomorphic on Δ except for the points on $|\Delta \cdot \Gamma|$. Now, let $\mathfrak{p}$ be a point on $|\Delta \cdot \Gamma|$. Then since Γ is a curve without singular points, we can choose the local coordinates z^1, z^2 in $\mathfrak{U}_\mathfrak{p}$ so that $R_\mathfrak{p} = z^1$ and that $z^2 = \tau_\mathfrak{p}$ on Γ. Put

$$N(z^1, z^2) = z^1 F = MR_\mathfrak{p}(D)R_\mathfrak{p}(\Delta).$$

Then, since $f \in \mathfrak{f}(\Gamma, D)$ implies that $M(0, z^2)R_\mathfrak{p}(D; 0, z^2) = f(z^2)R_\mathfrak{p}(D; 0, z^2)$ is holomorphic in z^2, we have

$$N(0, z^2) \equiv 0 (R_\mathfrak{p}(\Delta; 0, z^2)).$$

This shows that

$$N(z^1, z^2) \equiv 0 (z^1, R_\mathfrak{p}(\Delta; z^1, z^2)).$$

Now, let $\Delta^{(1)}$ be any irreducible branch of Δ passing through $\mathfrak{p}$ and $z^\alpha = \psi^\alpha(\tau_\Delta)$ $(\alpha = 1, 2)$ be a parametric representation of $\Delta^{(1)}$. Then we get from the above result

$$N(\psi^1(\tau_\Delta), \psi^2(\tau_\Delta)) \equiv 0 (\psi^1(\tau_\Delta)),$$

proving that $F_\Delta = F(\psi(\tau_\Delta)) = N(\psi(\tau_\Delta))/\psi^1(\tau_\Delta)$ is a holomorphic function of the local uniformization variable τ_Δ on $\Delta^{(1)}$. Thus F_Δ is holomorphic everywhere on Δ and therefore F_Δ must be constant on each irreducible component of Δ, while, since by hypothesis Δ is connected in $\mathfrak{M} - \Gamma$, any two irreducible components of Δ have at least one common point in $\mathfrak{M} - \Gamma$. Hence F_Δ reduces to a constant on Δ: $F_\Delta = c$. Now, putting $F_1 = F - c$, we get a meromorphic function F_1 satisfying $(F_1) \geqq \Delta - \Gamma$ and $R_\mathfrak{p}F_1/h_\mathfrak{p} = f$. The

corresponding meromorphic function $M_1 = F_1/F_0$ belongs clearly to $\mathfrak{F}(D)$ and satisfies $M_1(\phi_\mathfrak{p}) = f$. Thus *the mapping* $M \to f = M(\phi_\mathfrak{p})$ *gives the one-to-one correspondence between meromorphic functions* $M \in \mathfrak{F}(D)$ *and meromorphic functions* $f \in \mathfrak{f}(\Gamma, D)$ *satisfying* $Q_f[\Omega \cdot A_\lambda] = 0$ $(\lambda = 1, 2, \cdots, q)$.

Let j_D be the number of linearly independent simple differentials A of the first kind satisfying $Q_f[\Omega \cdot A] = 0$ for all $f \in \mathfrak{f}(\Gamma, D)$. Then the number of linearly independent conditions imposed on $f \in \mathfrak{f}(\Gamma, D)$ by $Q_f[\Omega \cdot A_\lambda] = 0$ $(\lambda = 1, 2, \cdots, q)$ is $q - j_D$ and therefore, by virtue of the above results,

$$\operatorname{def}(D/\Gamma) = \dim \mathfrak{f}(\Gamma, D) - \dim \mathfrak{F}(D) = q - j_D.$$

Combined with (7. 2), this proves the following

THEOREM 7. 1. *Assume* Γ *to be an irreducible curve without singular point. Let* D *be an arbitrary divisor not containing* Γ *as its component such that* $\Gamma - D$ *is linearly equivalent to a curve* Δ *which does not contain* Γ *as its component and is connected in* $\mathfrak{M} - \Gamma$, *and let* j_D *be the number of linearly independent simple differentials* A *of the first kind satisfying*

$$\sum_{\mathfrak{p} \in |E \cdot \Gamma|} f h_\mathfrak{p}(\mathfrak{p}) \operatorname{Res}_\mathfrak{p}[A_E/R_\mathfrak{p}] = 0, \qquad \text{for all } f \in \mathfrak{f}(\Gamma, D),$$

where E *is an arbitrary hyperplane section of* $\mathfrak{M}$ *cutting out on* Γ *only simple intersection points. Then we have*

$$\operatorname{def}(D/\Gamma) = q - j_D. \tag{7. 3}$$

From this theorem it follows immediately the following

THEOREM 7. 2. *Let* Γ *be an irreducible curve without singular points and* D *be an arbitrary divisor not containing* Γ *as its component. If* $\Gamma - D - E$ *is linearly equivalent to a curve* Δ *not containing* Γ *as its component, then we have* $\operatorname{def}(D/\Gamma) = 0$.

Proof. We choose E so that E cuts out on Γ only simple intersection points which do not lie on $|D|$ nor on $|\Delta|$ and that $\Delta + E$ is connected in $\mathfrak{M} - \Gamma$. Since Γ is linearly equivalent to $D + \Delta + E$, we may assume that

$$h_\mathfrak{p} = R_\mathfrak{p}(D; \phi_\mathfrak{p})\ R_\mathfrak{p}(\Delta; \phi_\mathfrak{p})\ R_\mathfrak{p}(E; \phi_\mathfrak{p}). \tag{7. 4}$$

Now, for each $\mathfrak{p} \in |E \cdot \Gamma|$, every $f \in \mathfrak{f}(\Gamma, D)$ is holomorphic at $\mathfrak{p}$, while (7. 4) shows that $h_\mathfrak{p}(\mathfrak{p}) = 0$. Hence we have, for arbitrary A,

$$\sum_{\mathfrak{p} \in |E \cdot \Gamma|} f h_\mathfrak{p}(\mathfrak{p}) \operatorname{Res}_\mathfrak{p}[A_E/R_\mathfrak{p}] = 0, \qquad \text{for all } f \in \mathfrak{f}(\Gamma, D).$$

Consequently $j_D = q$ and therefore, by (7. 3), $\operatorname{def}(D/\Gamma)$ is zero.

For an arbitrary given divisor D, the divisor class $\{D+nE\}$ contains an irreducible curve without singularities if the positive integer n is chosen sufficiently large.[55] Using this fact, we can deduce from Theorem 7. 2 immediately the following

THEOREM 7. 3 (O. Goldman [56]). *Let L, D be two arbitrary given divisors and Γ_n be an irreducible curve without singularities belonging to the divisor class $\{L+nE\}$, E being the hyperplane section. Then, if n is sufficiently large, we have* $\mathrm{def}(D/\Gamma_n)=0$.

The *superabundance* of an arbitrary divisor class $\{D\}$ is, by definition,

$$\sup\{D\}=\dim\{D\}+\dim\{K-D\}-I(D,D)+\pi_D-g+q-2,$$

where π_D is the *virtual genus* of D defined by $2\pi_D-2=I(D,D)+I(K,D)$. The *theorem of Riemann-Roch* on algebraic surfaces asserts that $\sup\{D\}$ *is always non-negative.* The divisor class $\{D\}$ is called *regular* if $\sup\{D\}$ and $\dim\{K-D\}$ are both zero.

Let D is an arbitrary divisor. Then, if n is sufficiently large, $\{D+nE-K\}$ is a divisor class of an irreducible curve Δ and therefore $\{D+nE\}=\{K+\Delta\}$; moreover, it is obvious that $\dim\{\Delta\}\geqq 2$ for sufficiently large n. Hence, by Theorem 6. 6, we get the following

THEOREM 7. 4 (G. Castelnuovo [57]). *For an arbitrary given divisor D, the divisor class $\{D+nE\}$ is regular if n is sufficiently large.*

Choose n so large that $\{D+nE\}$ is regular and contains a curve Γ. Then, using (5. 17), we get immediately $chd\{\Gamma\}=q$. Combined with (3. 35), this proves a theorem of G. Castelnuovo [58] to the effect that *the characteristic deficiency of a curve is always not greater than the irregularity q of the surface and there exists a curve having the characteristic deficiency q.* Again, combined with Theorem 3. 4, the above result yields the following

THEOREM 7. 5.[59] *Let Γ_n be a curve in the divisor class $\{D+nE\}$, D being an arbitrary fixed divisor. Then, if n is sufficiently large, there exists on $\mathfrak{M}$ exactly $2q$ independent simple Picard integrals of the second kind which are multiples of $-\Gamma_n$.*

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[55] See Goldman [9], § 5.
[56] Goldman [9], § 6.
[57] Castelnuovo [4]; Goldman [9], § 7.
[58] Castelnuovo [4]; see also Zariski [30], pp. 69-70.
[59] Cf. Zariski [30], pp. 122-128.

As was shown by O. Goldman,[60] *the theorem of Riemann-Roch on algebraic surfaces can be readily deduced from Theorem 7. 3 and Theorem 7. 4.* We shall reproduce here his deduction briefly. Let D be a given divisor and Γ be an irreducible curve without singularities belonging to the divisor class $\{nE\}$, n being a sufficiently large positive integer. Then we have, by Theorem 7. 3, $\operatorname{def}(K - D/\Gamma) = 0$ and therefore

$$\dim\{K - D\} = \dim_\Gamma\{K \cdot \Gamma - D \cdot \Gamma\},$$

where $\dim_\Gamma\{K \cdot \Gamma - D \cdot \Gamma\}$ denotes the dimension of the intersection divisor $K \cdot \Gamma - D \cdot \Gamma$ on the curve Γ. Again, since $\Gamma \cdot \Gamma + K \cdot \Gamma$ is a canonical divisor on Γ, we have, by the theorem of Riemann-Roch on the curve,

$$\dim_\Gamma\{D \cdot \Gamma + \Gamma \cdot \Gamma\} = I(\Gamma, D + \Gamma) - \pi_\Gamma + \dim_\Gamma\{K \cdot \Gamma - D \cdot \Gamma\} + 1,$$

while, it is obvious that

$$\dim_\Gamma\{D \cdot \Gamma + \Gamma \cdot \Gamma\} \geqq \dim\{D + \Gamma\} - \dim\{D\}.$$

Consequently we get

$$\dim\{D\} + \dim\{K - D\} \geqq \dim\{D + \Gamma\} - I(\Gamma, D + \Gamma) + \pi_\Gamma - 1.$$

On the other hand, Theorem 7. 4 shows that

$$\dim\{D + \Gamma\} = I(D + \Gamma, D + \Gamma) - \pi_{D+\Gamma} + g - q + 2.$$

Hence we get

$$\dim\{D\} + \dim\{K - D\} \geqq I(D, D + \Gamma) - \pi_{D+\Gamma} + \pi_\Gamma + g - q + 1,$$

yielding immeditely

$$\dim\{D\} + \dim\{K - D\} \geqq I(D, D) - \pi_D + g - q + 2,$$

q. e. d.

Remark. The arithmetic genus of an algebraic surface $\mathfrak{M}$ is, by definition, the constant p_a appearing in the "postulation formula"

$$\dim\{nE\} = c_0 n^2 + c_1 n + p_a + 1, \qquad \text{(for large } n\text{)},$$

where E denotes a hyperplane section of $\mathfrak{M}$ and c_0, c_1 are the constants not depending on n. On the other hand, it follows from Theorem 7. 4 that, for sufficiently large n, $\dim\{nE\}$ is given by

$$\dim\{nE\} = \tfrac{1}{2} I(E, E) n^2 - \tfrac{1}{2} I(K, E) n + g - q + 1.$$

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[60] Goldman [9], § 7.

Comparing this with the postulation formula, we infer immediately that *the difference $g - p_a$ between the geometric genus g and the arithmetic genus p_a of an algebraic surface $\mathfrak{M}$ is equal to the number q of linearly independent simple differentials of the first kind attached to $\mathfrak{M}$.*[61]

THE JOHNS HOPKINS UNIVERSITY.

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GAUSSIAN CURVATURE AND LOCAL EMBEDDING.*

By PHILIP HARTMAN ** and AUREL WINTNER

1. The embedding theorems. As recently shown in [2], [3], [4], the usual assumptions of differentiability in some standard theorems of differential geometry can be lightened considerably if considerations involving the Gauss formula (and its variants) for the surface curvature K are replaced by others involving the "curvatura integra," $\int\int K d\sigma$; cf. [8], pp. 43-44, [6], p. 135 and § 2 below. The present paper develops these ideas farther, with applications to the embedding problem.

A function of one or several variables will be called of class $C^n(\lambda)$, where $n = 0, 1, \cdots$ and $0 < \lambda < 1$, on a domain if the function is of class C^n on that domain (that is, if it possesses there continuous n-th partial derivatives) and if all the n-th derivatives satisfy a uniform Hölder condition of order λ (with respect to all of the variables, and on every compact subset of the domain). Let $(g_{ik}) = (g_{ik}(u, v))$, where $i, k = 1, 2$, be a symmetric positive-definite matrix defined on some (u, v)-domain. The line element

$$ds^2 = g_{ik}du^i du^k, \qquad \text{where } (u^1, u^2) = (u, v), \tag{1}$$

will be said to be of class C^n or of class $C^n(\lambda)$ if each coefficient function in (1) belongs to the class C^n or $C^n(\lambda)$, respectively. For $n \geqq 1$, the (g_{ik}) or (1) will be said to possess a local embedding of class C^n or $C^n(\lambda)$ at a given point (u_0, v_0), if, in a vicinity of (u_0, v_0), there exist three functions $x = x(u, v)$, $y = y(u, v)$, $z = z(w, v)$ of class C^n or $C^n(\lambda)$, respectively, satisfying

$$dx^2 + dy^2 + dz^2 = g_{ik}du^i du^k; \tag{2}$$

that is, if there exists, in the 3-dimensional Euclidean space, a 2-dimensional surface, of class C^n or $C^n(\lambda)$, on which (1) is the squared element of arc-length.

Several extensions of the theorem of [2], p. 563, will be proved. In the elliptic case, the result will be as follows:

(I) *Let* $0 < \mu < \lambda < 1$. *In a neighborhood of* $(u, v) = (0, 0)$, *let* (1)

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** John Simon Guggenheim Memorial Foundation Fellow, on leave of absence from The Johns Hopkins University.

be a positive-definite line element of class $C^1(\lambda)$ *which possesses a positive curvature* $K = K(u, v)$ *of class* $C^0(\lambda)$. *Then* (1) *has a local embedding of class* $C^2(\mu)$.

The analogue of (I), in which $C^0(\lambda)$, $C^1(\lambda)$, $C^2(\mu)$ are replaced by $C^{n-1}(\lambda)$, $C^n(\lambda)$, $C^{n+1}(\mu)$, respectively, where $n \geqq 2$, was proved in [2]. The definition of the curvature K of (1), when (1) is of class C^1, will be given in § 2 below, following the ideas mentioned at the beginning of § 1. Whether or not the $C^2(\mu)$ in the assertion of (I) can be improved to $C^2(\lambda)$ will remain undecided. Also, it will be left open whether or not the Hölder conditions can be dropped in both the assumption and assertion of (I). It will, however, be proved that they can be omitted in the hyperbolic analogue of (I):

(II) *In a neighborhood of* $(u, v) = (0, 0)$, *let* (1) *be a positive-definite line element, of class* C^1, *which possesses a continwous negative curvature* $K = K(u, v)$. *Then* (1) *has a local embedding of class* C^2.

Since the surface (x, y, z) of class C^2 supplied by (II) has a continuous Gaussian curvature which is identical with K, the assumption in (II) that (1) possess a continuous curvature is necessary. In this regard, cf. the remarks in [2], p. 554 or the footnote in [4], p. 144.

Corresponding to (II), the Hölder conditions can be omitted in the hyperbolic case of the theorem of [2], p. 563, as follows:

(III) *Let* $n \geqq 1$. *In a neighborhood of* $(u, v) = (0, 0)$, *let* (1) *be a positive-definite line element, of class* C^n, *which posseses a negative curvature of class* C^{n-1}. *Then* (1) *has a local embedding of class* C^{n+1}.

It remains undecided whether the assertion of (III) does or does not remain true if the curvature, instead of being negative, is positive.

2. The generalized definition of curvature. Let (1) be a positive-definite line element on a simply connected (u, v)-domain R. If (1) is of class C^2, the curvature $K = K(u, v)$ of (1) can be defined by the standard formula

$$K = (B_u - A_v)/g, \tag{3}$$

where $g = \det^{\frac{1}{2}}(g_{ik}) > 0$ and

$$A = \tfrac{1}{2}g^{-1}[(g_{12}/g_{11})g_{11u} + g_{11v} - g_{12u}], \tag{4}$$

$$B = \tfrac{1}{2}g^{-1}[(g_{12}/g_{11})g_{11v} - g_{22u} + g_{12v}]; \tag{5}$$

cf., e. g., [1], p. 216. If J is a Jordan curve in R and is piecewise of class C^1, and if T is the interior of J, then (3) and Green's theorem imply that

$$\phi(T) = \int_J A\,du + B\,dv, \tag{6}$$

where the set-function $\phi(T)$ is the "curvatura integra,"

$$\phi(T) = \iint_T K\,d\sigma, \qquad (d\sigma = g\,du\,dv). \tag{7}$$

In what follows, by "an admissible Jordan domain T" will be meant a domain T of the type admitted in (6) and (7). The formula (6) will be used as a basis for the following definition:

Definition. If (1) is a positive-definite line element of class C^1 (not necessarily C^2) on a simply-connected domain R, the "curvatura integra," $\phi(T)$, of (g_{ik}) or (1) is defined for admissible Jordan domains T by (6), where A, B are given by (4), (5). If, in addition, the set-function $\phi(T)$ is absolutely continuous, that is, if $\phi(T)$ has a representation of the form (7) in terms of a function $K = K(u, v)$ which is independent of T and which is summable (i. e., class L_1) over every compact subset of R, then (g_{ik}) or (1) will be said to possess the *curvature* $K = K(u, v)$.

In this definition, the function $K(u, v)$, if it exists, is only determined almost everywhere (that is, up to a 2-dimensional zero set in R). But even such a determination of a curvature can be useful; cf., e. g., the proof of Theorem III in [4] (where the assumptions imply, however, that K exists and is bounded almost everywhere on compact subsets of R).

The following standard conventions concerning K will be made below: If there exists a continuous function $K = K(u, v)$ satisfying (6) and (7), then the curvature (1) will be considered to be defined everywhere (instead of almost everywhere) as the unique continuous K; in which case (1) will be said to possess a curvature of class C^0. Corresponding remarks apply to a curvature of class C^n or $C^n(\lambda)$.

Similar extensions of the classical definition of the curvatura integra or of the curvature have been given by replacing the classical formula (3) by the formula of Gauss-Bonnet ([8], pp. 43-44; also [6], p. 135) or by the Frobenius formula for K ([2], p. 562; [3], p. 759). These interpretations are equivalent, as seen from the case $\Gamma \equiv \Gamma^m \equiv 1$ of theorem (i) below (they are clearly equivalent for "smooth" line elements).

It may be mentioned that the equivalence of the two definitions of the

curvatura integra, those based on (3) and the formula of Gauss-Bonnet, respectively, leads to a simple geometrical interpretation of the set-function $\phi(T)$, even for line elements (1) which are just of class C^1. In fact, $\phi(T)$ is just the oriented variation of the direction of a vector transported parallel to itself (Levi-Civita) along the boundary, J, of T. In order to see this, it is sufficient to write the formula of Gauss-Bonnet in the form

$$\Sigma\alpha + \Sigma\Delta\theta + \phi(T) = 2\pi,$$

where $\Sigma\alpha$ is the sum of the oriented exterior angles α (measured in the metric (1)) at the vertices of J, the sum $\Sigma\Delta\theta$ is extended over the sub-arcs of J, all of which are supposed to be of class C^1, and $\Delta\theta$ denotes the oriented variation of a continuous determination of θ along one of the sub-arcs, finally $-\theta$ is the oriented angle from the tangent vector of the sub-arc to a vector transported by parallelism along the sub-arc; cf. the proof of the Gauss-Bonnet formula in [6], p. 134, and, for the above C^1-formulation, a remark in [5], § 15.

In applications of the set-function $\phi(T)$, the following fact will be needed:

(i) *Let the function* $\Gamma = \Gamma(u, v)$ *and the positive-definite line element* (1) *be of class* C^1 *on a simply-connected domain* R. *For* $m = 1, 2, \cdots$, *let the functions* $\Gamma^m = \Gamma^m(u, v)$ *and the positive-definite line elements*

$$(1^m) \qquad ds^2 = g^m{}_{ik}du^i du^k \qquad (w^1 = u, w^2 = v)$$

be of class C^2 *on* R *and have the property that, as* $m \to \infty$, *the functions* Γ^m, $g^m{}_{ik}$ *and their first order partial derivatives tend to the functions* Γ, g_{ik} *and their first order partial derivatives, respectively, uniformly on every compact subset of* R. *Let* T *be an admissible Jordan domain in* R. *Then*

$$(8) \qquad \lim_{m\to\infty} \iint_T \Gamma^m K^m d\sigma^m$$

exists and has the value

$$(9) \qquad \int_J \Gamma(A du + B dv) + \iint_T (A\Gamma_v - B\Gamma_u) du dv.$$

If, in addition, (1) *has a curvature* K, *then the limit in* (8) *has the value*

$$(10) \qquad \int_T \Gamma K d\sigma.$$

As a consequence of (8), (9) for the case $\Gamma \equiv \Gamma^m \equiv 1$, the relation

$$\lim_{m\to\infty} \phi^m(T) = \phi(T) \tag{11}$$

holds for every admissible Jordan domain T in R.

If $f(u)$, $f^m(u)$ are positive functions of class C^2, say for $-1 \leqq u \leqq 1$, such that, as $m \to \infty$, both $f^m \to f$ and $f^m{}_u \to f_u$ hold uniformly in u, while $f^m{}_{uu} \to f_{uu}$ does not hold, then it is seen, by considering the cases $ds^2 = du^2 + (f(u))^2 dv$ and $ds^2 = du^2 + (f^m(u))^2 dv^2$ of (1) and (1^m), respectively, that $K^m \to K$ need not hold (even if K is continuous). Thus a "direct" verification of the equality of the integral (10) and the limit (8) is not possible.

Proof of (i). It is clear from (6), (7) and Green's theorem that the integral in (8) is equal to the expression (9^m), if (9^m) belongs to Γ^m and (1^m) as (9) does to Γ and (1). Thus it is clear that the limit (8) exists and has the value (9).

If $K = K(u, v)$ is continuous, instead of being merely summable, the equality of the expressions (9) and (10) is an immediate consequence of the lemma in [3], p. 761, since (6) and (7) hold for arbitrary admissible Jordan domains T. However, it is easy to see that the proof of that lemma in [3] can be modified so as to apply to the case of a summable K. Thus, (i) is proved.

(ii) *Let* (1) *be a positive-definite line element of class* C^1 *on a simply-connected domain* R, *and let* (1) *possess a curvature* $K = K(u, v)$. *Let a function* $z = z(u, v)$ *of class* C^2 *on* R *be such that the line element defined by*

$$\gamma_{ik} du^i du^k = g_{ik} du^i du^k - (dz)^2 \tag{12}$$

is positive-definite on R. *Then* (12) *possesses a curvature* $\kappa = \kappa(u, v)$ *satisfying*

$$\kappa\gamma^4 = -g(rs - t^2) + L_1 r + L_2 s + L_3 t + Q + Kg^4. \tag{13}$$

Here γ *and* p, q, r, s, t *denote* $\det^{\frac{1}{2}}(\gamma_{ik}) > 0$ *and the partial derivatives* z_u, z_v, z_{uu}, z_{uv}, z_{vv} *of* z, *respectively*; L_1, L_2, L_3 *are linear forms in* p, q, *while* Q *is a quadratic form in* p, q, *and these forms have coefficients which are quadratic forms in the* g_{ik} *and their first order partial derivatives.*

Proof of (ii). If (1) is of class C^2 and if the partial differentiations, occurring in (3), are carried out, it is seen that the formula (3) for K can be put into the form

$$Kg^4 = \tfrac{1}{2} g^2 [2g_{12uv} - g_{11vv} - g_{22uu}] + b_1 g_{11} + b_2 g_{12} + b_3 g_{22},$$

where b_1, b_2, b_3 are quadratic forms in the first order partial derivatives of the g_{ik}; cf. the standard Frobenius formula for K, [2], p. 553. If, in addition, $z(u, v)$ is of class C^3, then (12) is of class C^2 and possesses, therefore, a curvature K. The function $\kappa\gamma^4$ can be calculated by replacing g_{11}, g_{12}, g_{22} in the last formula line by $\gamma_{11} = g_{11} - p^2$, $\gamma_{12} = g_{12} - pq$, $\gamma_{22} = g_{22} - q^2$, respectively. The result of this calculation turns out to be (13); cf. [1], p. 254, where this calculation is carried out in the main. The expression on the right-hand side of (13) is identical with 4 times the expression on the left-hand side of equation (4) in [1], p. 254, except that those terms of the latter equation which do not contain p, q, r, s, t have been grouped together as Kg^4 in (13) above.

It remains to consider the case in which (1) and $z(u, v)$ are of class C^1, C^2, respectively. Let (1^m) be the m-th element of a sequence of positive-definite line elements satisfying the conditions of (i). Let $z^m = z^m(u, v)$, where $m = 1, 2, \cdots$, be a sequence of functions of class C^3 on R, and suppose z^m and its first and second order partial derivatives tend, as $m \to \infty$, to z and its corresponding derivatives, respectively, uniformly on every compact subset of R. Let $\gamma^m{}_{ik}(u, v)$ be defined by

$$(14^m) \qquad \gamma^m{}_{ik}du^i du^k = g^m{}_{ik}du^i du^k - (dz^m)^2.$$

Corresponding to any compact subset of R, the line element (14^m) is positive-definite for sufficiently large m. On those subdomains of R on which (14^m) is positive-definite, let κ^m be the curvature of (14^m), and let $d\sigma^m = \gamma^m dudv$, where $\gamma^m = \det(\gamma^m{}_{ik}) > 0$.

Let T be an admissible Jordan domain in R. Then the formula for κ^m, deduced at the beginning of this proof, and the case $\Gamma = g^4\gamma^{-3}$, $\Gamma^m = (g^m)^4(\gamma^m)^{-3}$ of (i) imply that $\lim \iint_T \kappa^m d\sigma^m$, as $m \to \infty$, exists and has the value

$$\iint_T \{-g^2(rs - t^2) + L_1 r + L_2 s + L_3 t + Q\}\gamma^{-3} dudv + \iint_T Kg^4\gamma^{-3} dudv.$$

Hence, the last expression is the "curvatura integra" of (12); cf. (11). Consequently, the above Definition of curvature shows that (12) has a curvature and that (13) holds. This completes the proof of (ii).

3. The case $K \equiv 0$. If a positive-definite line element (1) is of class C^2 and has a curvature $K \equiv 0$ on a simply-connected domain R, then a

standard result of differential geometry states that there exist functions $x = x(u, v)$, $y = y(u, v)$ of class C^2 on R such that

$$g_{ik}du^i dw^k = (dx)^2 + (dy)^2; \tag{15}$$

cf., e. g., [1], pp. 221-222. In [3], p. 767, this fact was extended to the case in which C^2 is replaced by C^1 in both the assumption and assertion. However, as shown by the following theorem, much more can be proved under the mere C^1-assumption:

(iii) *If a positive-definite line element* (1) *of class* C^1 *possesses a curvature* $K \equiv 0$ *on a simply-connected domain* R, *then there exist functions* $x = x(u, v)$, $y = y(u, v)$ *of class* C^2 *on* R *satisfying* (15). *Furthermore, such functions* x, y *can be obtained by quadratures.*

The proof will follow that given in [1], pp. 221-222, for the case in which (1) is of class C^2. It will show that if (1) if of class C^n, where $n \geqq 1$, then x and y can be chosen of class C^{n+1}.

Proof of (iii). It is seen from (7) that if $K \equiv 0$, the line integral in (6) is 0 for all Jordan curves J in R which are piecewise of class C^1. Hence, there exist functions $\theta = \theta(u, v)$ of class C^1 in R satisfying

$$\theta_u = A \quad \text{and} \quad \theta_v = B. \tag{16}$$

Such a θ can be obtained by quadratures and is determined by (16) to an additive constant.

It follows from (16) and the definitions, (4) and (5), of A and B, that

$$(g_{11}^{\frac{1}{2}} \cos\theta)_v = (g_{12}g_{11}^{-\frac{1}{2}} \cos\theta + g g_{11}^{-\frac{1}{2}} \sin\theta)_u$$

and

$$(g_{11}^{\frac{1}{2}} \sin\theta)_v = (g_{12}g_{11}^{-\frac{1}{2}} \sin\theta - g g_{11}^{-\frac{1}{2}} \cos\theta)_u.$$

Hence there exist on R functions $x = x(u, v)$, $y = y(u, v)$ of class C^2 satisfying

$$x_u = g_{11}^{\frac{1}{2}} \cos\theta, \; x_v = g_{12}g_{11}^{-\frac{1}{2}} \cos\theta + g g_{11}^{-\frac{1}{2}} \sin\theta$$

and

$$y_u = g_{11}^{\frac{1}{2}} \sin\theta, \; y_v = g_{12}g_{11}^{-\frac{1}{2}} \sin\theta - g g_{11}^{-\frac{1}{2}} \cos\theta,$$

and these functions can be obtained by quadratures. The relation (15), to be proved, is an immediate consequence of the last two formula lines. This proves (iii).

In the proofs of (I) and (II), theorem (iii) will allow the avoidance of

the conformal normal form for (12), used in [2], pp. 555, 561, 562. This will permit the omission of the Hölder condition in (II).

4. *Proof of* (I). Following a device of Weingarten ([7], pp. 3-4; cf. [1], pp. 253-254), let a function $z = z(u, v)$ be sought which is of class C^2 in a vicinity of $(u, v) = (0, 0)$ and makes (12) a positive-definite line element with curvature $K \equiv 0$. In view of (ii), this leads to a Monge-Ampère partial differential equation, obtained by equating to 0 the right-hand side of (13).

The smoothness assumptions of (I) show that in this equation the coefficients, which are known functions of (u, v), are of class $C^0(\lambda)$. The assumption $K > 0$ means that the Monge-Ampère partial differential equation is of elliptic type; cf. the assumptions of the Theorems, p. 554 and the lemmas on pp. 556-557 of [2]. Hence, Lemma 2 in [2], p. 557, supplies the existence of solutions $z = z(u, v)$ of class $C^2(\mu)$ in a sufficiently small neighborhood of $(u, v) = (0, 0)$. The remarks on pp. 560-561 in [2] show that solutions z can be so chosen that (12) becomes positive-definite in a vicinity of $(0, 0)$.

If $z = z(u, v)$ denotes such a solution, then (iii) states that there exist functions $x = x(u, v)$, $y = y(u, v)$ of class C^2 in a vicinity of $(0, 0)$ satisfying $\gamma_{ik} du^i du^k = (dx)^2 + (dy)^2$. Actually, the proof of (iii) shows that x, y can be chosen of class $C^2(\mu)$, since the γ_{ik} are of class $C^1(\mu)$. In view of (12), the functions x, y, z form a solution of the system of partial differential equations (2). This completes the proof of (I).

5. *Proof of* (II) *and* (III). It is clear from the preceding proof that, in order to prove (II), it is sufficient to verify, under the conditions of (II), the following fact: The Monge-Ampère equation obtained by equating to 0 the right-hand side of (13), has, in a vicinity of $(u, v) = (0, 0)$, solutions $z = z(x, y)$ of class C^2 which make (12) positive-definite. The smoothness assumptions of (II) mean that the known coefficient functions of (u, v) in this Monge-Ampère equation are continuous near $(u, v) = (0, 0)$, while the hypothesis $K < 0$ implies that this equation is of hyperbolic type.

The existence statement supplied by Lemma 2 in [2] is not directly applicable. However, it is easy to see that, *in the hyperbolic case*, the proof of Lemma 2 (and that of Lemma 1, on which Lemma 2 depends) remains valid if the Hölder conditions occurring there are ignored. In fact, in the proofs on pp. 556-560 in [2], it is only necessary to interpret the symbol $|f|_\mu$ as the maximum of $|f|$ on the circle $u^2 + v^2 \leqq r^2$ and, as before, $\| w \|$ as

greatest of the six numbers $|w|_\mu$, $|w_u|_\mu$, $|w_v|_\mu$, $|w_{uu}|_\mu$, $|w_{uv}|_\mu$, $|w_{vv}|_\mu$. This proves (II).

The remark following Lemma 2 in [2] remains valid if the Hölder conditions there are ignored. This means that if the coefficient functions in the hyperbolic Monge-Ampère equation are of class C^{n-1} (as under the assumptions of (III)), then the solutions $z = z(u, v)$ furnished by the above existence proof are of class C^{n+1}. In addition, the remark following the statement of (iii), above, shows that the corresponding functions $x = x(u, v)$ and $y = y(u, v)$ are of class C^{n+1}. This proves (III).

PARIS, FRANCE AND THE JOHNS HOPKINS UNIVERSITY.

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ON AN OSCILLATION CRITERION OF LIAPOUNOFF.*

By PHILIP HARTMAN ** and AUREL WINTNER

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1. In the differential equation

$$x'' + q(t)x = 0, \tag{1}$$

let $q(t)$ be a real-valued, continuous function on a closed interval, say on $[0, T]$. Consider only the real-valued, non-trivial ($\not\equiv 0$) solutions $x(t)$ of (1). If no such solution has more than one zero on $[0, T]$, let (1) be called disconjugate on $[0, T]$. A general approach to disconjugate equations (1) was considered in [3]; it easily leads to various criteria of explicit nature, in particuar to the following criterion, which goes back to Liapounoff (cf. [1]): If

$$T\int_0^T |\, q(t)\,|\, dt \leqq 4, \tag{2}$$

then (1) is disconjugate on $[0, T]$. It is also known ([2]; cf. also [1]) that the 4 in (2) cannot be improved to any $4 + \epsilon > 4$ (if $q(t)$ is unspecified). But it will turn out below that (2) can greatly be improved in another direction.

First, if the factor T in (2) is written beneath the integral sign and is then diminshed to t, where $0 \leqq t \leqq T$, it is natural to ask whether there exists a positive absolute constant having the property that (1) must be disconjugate on $[0, T]$ whenever the value of

$$\int_0^T t\,|\, q(t)\,|\, dt \tag{3}$$

does not exceed that absolute constant. It turns out that such an absolute constant exists, and that its best value is 1. In other words, (1) must or need not be disconjugate on $[0, T]$ according as the value of (3) does not or does exceed 1.

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** John Simon Guggenheim Memorial Foundation Fellow, on leave of absence from The Johns Hopkins University.

It will also follow that if (3) is replaced by

$$\int_0^T t^a \,|\, q(t) \,|\, dt, \tag{4}$$

then there does not exist for any $T > 0$ and for any $a > 1$ a positive $C = C(a, T)$ having the property that (1) must be disconjugate on $[0, T]$ whenever the value of (4) is less than $C(a, T)$.

2. Clearly, neither is the sufficiency of

$$\int_0^T t \,|\, q(t) \,|\, dt \leqq 1 \tag{5}$$

contained in that of (2), nor is the converse true. But the sufficiency of both (2) and (5) is contained in part (i), and the final nature of the respective absolute constants, 4 and 1, in part (ii), of the following theorem:

Let $r = r(t)$ be a positive, continuous function on the open interval $(0, T)$.

(i) *Whenever $q(t)$ is a continuous function on the closed interval $[0, T]$ satisfying*

$$\int_{+0}^{T-0} r(t) \,|\, q(t) \,|\, dt \leqq T \operatorname*{g.\,l.\,b.}_{0<t<T} \{r(t)/t(T-t)\}, \tag{6}$$

the equation (1) must be disconjugate on $[0, T]$, but

(ii) *the assertion of (i) becomes false for every $r(t)$ (and for every T) if the bound on the right of (6) is increased by ϵ, where $\epsilon > 0$ is arbitrary.*

Clearly, (2) and (5) result by choosing $r(t) = 1$ and $r(t) = t$, respectively. More generally, let $r(t) = t^a$, where $a \geqq 0$. Then the integral on the left of (6) becomes (4), while the bound on the right of (6) has the value 0, 1 or $T^{a-1}(2-a)^{2-a}/(1-a)^{1-a}$ according as $a > 1$, $a = 1$ or $a < 1$. In the more symmetric case, where $r(t) = t^a$ is replaced by $r(t) = t^a(T-t)^a$, $0 \leqq t \leqq T$, the bound on the right of (6) has the value 0 or $4^{1-a}T^{2a-1}$ according $a > 1$ or $a \leqq 1$. Thus the assertion made after (4), according to which there cannot exist a $C = C(a, T) > 0$ if $a > 1$, is obvious.

3. Let (6⁺) denote the inequality which results if $|\, q(t) \,|$ in (6) is replaced by the (non-negative, continuous) function $q^+(t) = \max(0, q(t))$. Then (6) implies (6⁺), but not conversely. The assertion of (i) remains,

however, true if its assumption (6) is relaxed to (6⁺). But this refinement of (i) is contained in (i) itself. In order to see this, it is sufficient to apply (i), as it stands, to the differential equation $x'' + q^+(t)x = 0$. Then if (6⁺) holds, this differential equation is disconjugate on $[0, T]$. Hence (1) is also, since $q(t) \leqq q^+(t)$. In fact, if (1_1), (1_2) denote the differential equations (1) which belong to coefficient functions $q = q_1$, $q = q_2$ satisfying $q_1(t) \leqq q_2(t)$, then it is clear from Sturm's comparison theorem that (1_1) must be disconjugate on $[0, T]$ if (1_2) is.

This Sturmian argument also shows that it is sufficient to prove (i) for non-negative $q(t) = |q(t)|$. In addition, it can be assumed that $T = 1$ (as seen if t in (1) and (6) is replaced by t/T). But if $T = 1$ and $q(t) \geqq 0$, then (6) reduces to

$$\int_{+0}^{1-0} r(t)q(t)\,dt \leqq \gamma_r, \tag{8}$$

where

$$\gamma_r = \underset{0<t<1}{\text{g. l. b.}} \{r(t)/t(1-t)\} \tag{9}$$

and, by assumption,

$$q(t) \geqq 0 \text{ on } [0,1] \text{ and } r(t) > 0 \text{ on } (0,1). \tag{10}$$

Thus it is sufficient to prove (i)-(ii) for (1) under the normalizations (8)-(10).

The proof of (i) will lead to a generalization of an inequality of Beurling (cf. [1]), while (ii) will be proved by an adaptation of a method of Borg [1]. In principle, both (i) and (ii) must be deducible from the general criterion used in [3].

4. *Proof of* (i). If $\gamma_r = 0$, then (8) and (10) imply $q(t) \equiv 0$, and so (1) is disconjugate. Thus, it remains to consider the case $\gamma_r > 0$. It is clear from (8), (9) and (10) that

$$\int_{+0}^{1-0} r(t)q(t)\,dt \geqq \gamma_r \int_0^1 t(1-t)q(t)\,dt. \tag{11}$$

Hence, since $\gamma_r > 0$, it is sufficient to prove (i) for the case $r(t) = p(t)$, where $p(t) = t(1-t)$. In fact, (9) shows that $\gamma_p = 1$. Accordingly, (i) will be proved if it is shown that if $q(t) \geqq 0$, then

$$\int_0^1 t(1-t)q(t)\,dt > 1 \tag{12}$$

must hold whenever (1) possesses some solution $x(t) \not\equiv 0$ which has at least two zeros on $[0, 1]$.

Suppose that there exists such a solution $x(t)$, and let $t = a$ and $t = b$, where

$$0 \leqq a < b \leqq 1, \tag{13}$$

be two zeros of it. Since its zeros cannot cluster on $[0, 1]$, and since $x(t)$ can be replaced by $-x(t)$, it can be supposed that $x(t)$ is positive on the open interval (a, b). Then, by (1),

$$\int_a^b t(1-t)q(t)\,dt = \int_{a+0}^{b-0} -t(1-t)x''(t)/x(t)\,dt. \tag{14}$$

On the other hand, since $q(t) \geqq 0$, it is seen from (1) that the graph of $x = x(t)$ is a convex arch over (a, b). Since $x(a) = 0$ and $x(b) = 0$ (hence, $x(t) \equiv 0$ being excluded, $x'(a) \neq 0$ and $x'(b) \neq 0$), this convexity implies that

$$t/x(t) \geqq 1/x'(a) > 0 \text{ and } (1-t)/x(t) \geqq -1/x'(b) > 0, \tag{15}$$

if t is on the interval (a, b). Finally, if c is any point of this interval, then it is seen from (13) that $1 - t > 1 - c$ or $t > c$ according as t is on (a, c) or on (c, b). It follows therefore from (15) that the integral on the right of (14) is greater than (not equal to)

$$-(1-c)\int_a^c x''(t)\,dt/x'(a) + c\int_c^b x''(t)\,dt/x'(b). \tag{16}$$

Hence, this lower estimate holds for the integral on the left of (14).

This lower estimate is valid for every choice of c on (a, b). Let c now be so chosen that $x'(c) = 0$ (such a choice is possible, since $x(a) = 0$ and $x(b) = 0$). Then (16) reduces to

$$-(1-c)(0 - x'(a))/x'(a) + c(x'(b) - 0)/x'(b),$$

or simply to $(1 - c) + c = 1$. Hence, the lower estimate, mentioned after (16), becomes

$$\int_a^b t(1-t)q(t)\,dt > 1. \tag{17}$$

Since $q(t) \geqq 0$, it is seen from (13) that (17) implies (12).

This completes the proof of (i).

5. *Proof of* (ii). Assume the normalizations (8)-(10). Let s and ϵ be arbitrary numbers satisfying $0 < \epsilon < s < 1 - \epsilon$, and let $x(t)$, where $0 \leqq t \leqq 1$, be any function having the following properties: $x(t)$ has a continuous, non-positive second derivative on $[0, 1]$, is positive on $(0, 1)$, finally

$$x(t) = t \text{ on } [0, s-\epsilon] \text{ and } x(t) = (1-t)(s-\epsilon)/(1-s-\epsilon) \text{ on } [s+\epsilon, 1].$$

In terms of this $x(t) = x(t; s, \epsilon)$, define a continuous $q(t) = q(t; s, \epsilon)$ by (1), by placing $q = -x''/x$. Then $q(t)$ is non-negative on $[0, 1]$ and vanishes identically on both sub-intervals $[0, s-\epsilon]$, $[s+\epsilon, 1]$ of $[0, 1]$. Since, by the last formula line, the solution $x(t)$ of (1) has two zeros on $[0, 1]$ (but does not vanish identically), the assertion of (ii) will be proved if it is shown that the value of the integral on the left of (8) can be made less than any number which exceeds the value (9), if s and ϵ are suitably chosen.

The last formula line and (1) show that the integral on the left of (8) is identical with

$$\int_{s-\epsilon}^{s+\epsilon} -r(t)x''/x(t)\,dt. \tag{18}$$

But the value of (18) is less than $1/(s-\epsilon)$ times the value of the integral of $-r(t)x''(t)$ over the interval $[s-\epsilon, s+\epsilon]$, since $x(t) \geqq s-\epsilon$ on this interval. Hence, if $m = m(s, \epsilon)$ denotes the maximum of $r(t)$ on $[s-\epsilon, s+\epsilon]$, then the value of (18) is less than $m/(s-\epsilon)$ times

$$\int_{s-\epsilon}^{s+\epsilon} -x''(t)\,dt = x'(s-\epsilon) - x'(s+\epsilon).$$

But the last difference is identical with $1 + (s-\epsilon)/(1-s-\epsilon)$, as seen from the definition of $x(t)$. Consequently, the value of (18) is less than that of the ratio

$$m(1-2\epsilon)/\{(s-\epsilon)(1-s-\epsilon)\}, \text{ where } m = m(s, \epsilon). \tag{19}$$

Since $r(t)$ is continuous at $t = s$, it is clear from the definition of m that the ratio (19) can be brought arbitrarily close to $r(s)/\{s(1-s)\}$ by choosing ϵ small enough. Since s is an arbitrary point on $(0, 1)$, it follows that, if ϵ and s are suitably chosen, the value of (19) will come arbitrarily close to the constant (9). Consequently, the value of (18) can be made less than any number that is greater than the constant (9). This proves (ii).

PARIS, FRANCE AND THE JOHNS HOPKINS UNIVERSITY.

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BOOLEAN ALGEBRAS WITH OPERATORS.*

PART I.

By BJARNI JÓNSSON and ALFRED TARSKI.

Introduction.

In the last few years various new kinds of algebraic systems have been defined and studied which can be referred to collectively as *Boolean algebras with operators*. These new algebraic systems have been obtained by enriching Boolean algebras by means of some new operations which are assumed to be additive, i. e., distributive under ordinary Boolean addition. As examples closure algebras, projective algebras, and relation algebras [1] may be mentioned. Like most other algebraic systems which have been studied in modern algebra, all these systems have arisen as results of generalizations of specific models studied in various parts of mathematics. Whenever a new kind of algebraic system originates in this way, the problem presents itself whether the abstract characterization is adequate in the sense that every system under discussion is isomorphic to one of the original models; this is the so-called representation problem for the given class of algebraic systems. As is well known, this problem has been completely solved for Boolean algebras without operators by showing that every Boolean algebra is isomorphic to an algebra formed by a field of sets under the set-theoretical operations of addition (formation of unions) and multiplication (formation of intersections).[2] It is also well known that this representation theorem is a simple consequence of (and trivially equivalent to) the so-called extension theorem by which every Boolean algebra can be extended to a complete and atomistic Boolean algebra. The aim of this work is to extend these results to arbitrary Boolean algebras with operators. We first establish the *extension theorem* by showing that every

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[1] Axiomatic studies of closure algebras, projective algebras, and relation algebras were first undertaken in McKinsey-Tarski [1], Everett-Ulam [1], and Tarski [2], respectively. The operator lattices discussed in Duffin-Pate [1] constitute still another example of Boolean algebras with operators. The general notion of a Boolean algebra with operators was first introduced and the main result of the present work were briefly stated in Jónsson-Tarski [1] (the abstract of a paper read at a meeting of the American Mathematical Society in November 1947). The numbers in brackets refer to the bibliography which follows the introduction.

[2] Cf. Stone [2], Theorem 67.

algebra of this kind can be extended to a complete and atomistic Boolean algebra where, moreover, all the operations are completely additive, i. e., distributive under addition of infinitely many elements. We also show that every equation involving Boolean addition and multiplication and the operators, which is identically satisfied in the original algebra, is also identically satisfied in the extended algebra. Hence, if the original algebra belongs to a class of algebras characterized by means of postulates all of which are algebraic identities (as, e. g., in the case of closure algebras), then the extended algebra belongs to the same class. This extension theorem implies a general *representation theorem* for Boolean algebras with operators; roughly speaking every such algebra is isomorphic to an algebra formed by a field of sets with the usual set-theoretical operations, and with operators defined as images under certain relations between elements of the universal set (the largest set) of this field of sets, the notion of the image under a relation being a generalization of the notion of the image under a function. We can also express this by saying that every Boolean algebra with operators is isomorphic to a subalgebra of the complex algebra of an algebraic system formed by a set and certain relations between its elements. When applying these general results to special Boolean algebras with operators, e. g., to those mentioned at the beginning of this introduction, we do not always obtain the natural representation theorem suggested by the origin of these algebras. In some cases a natural representation theorem can be obtained by combining our results with some arguments known from the literature.

After deriving at the beginning of Section 1 certain rather elementary theorems concerning additive functions, we define the notion of conjugate functions and investigate its basic properties. In the last part of this section we introduce the notion of a perfect extension of a Boolean algebra and restate the extension theorem for Bolean algebras without operators in terms of this concept.

In Section 2 the extension theorem for Boolean algebras with operators is proved. This result is used in Section 3 to obtain the representation theorem for Boolean algebras with operators. In the last part of this section both theorems are applied to closure algebras and to cylindric algebras (which present a generalization of the projective algebras mentioned above).

In Section 4 the extension theorem is applied to relation algebras. From this we obtain a new kind of representation theorem for relation algebras which is not a special case of our general representation theorem and which in certain rather special cases proves to supply a natural representation for these algebras.

In the fifth and last section we study the complex algebras of Brandt groupoids and groups. Here we also introduce the notion of a generalized Brandt groupoid and show that a relation algebra has a natural representation if, and only if, it is isomorphic to a subalgebra of the complex algebra of a generalized Brandt groupoid.

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Notations.

Throughout this work the letters j, m, n, p, q, and r will denote non-negative integers. By a sequence we shall, unless otherwise specified, mean a finite sequence. If x is an m-termed sequence, then for $j < m$ we understand x_j to be the $(j+1)$-st term of x. If x is an m-termed sequence and y is an n-termed sequence, then $x^\frown y$ denotes the $(m+n)$-termed sequence z such that $z_j = x_j$ for $j < m$ and $z_j = y_{j-m}$ for $m \leq j < m+n$. No distinction is made between an element a and the one-termed sequence x such that $x_0 = a$. Thus, given two elements a and b, we write alternatively $a^\frown b$ or $\langle a, b\rangle$ for the two-termed sequence, or ordered couple, x such that $x_0 = a$ and $x_1 = b$. If X is an m-termed sequence of sets, then $X_0 \times X_1 \times \cdots \times X_{m-1}$ denotes the set of all m-termed sequences x such that $x_j \in X_j$ (x_j is in X_j) for $j < m$; in case $X_j = A$ for every $j < m$, we agree that $A^m = X_0 \times X_1 \times \cdots \times X_{m-1}$.

A family A of sets is called a set-field if the union $X \cup Y$ and the difference $X - Y$ of any two sets X and Y in A are themselves members of A and if there exists a largest set in A, i. e., a set $U \in \mathsf{A}$ such that $X \subseteq U$ whenever $X \in \mathsf{A}$. This set U can be defined as the union of all sets in A:

$$U = \bigcup_{X \in \mathsf{A}} X;$$

it is called the universal set of A. From this definition it follows that the intersection $X \cap Y$ of any two members of A is itself a member of A and that the empty set Λ belongs to A.

As is well known, m-ary relations can be thought of as sets of m-termed sequences. We can therefore apply to them the usual set-theoretical operations. In the case of binary relations we shall need two other operations, namely relative multiplication and conversion. The *relative product* of two binary relations R and S, denoted by $R|S$, is the set of all ordered couples $\langle x, y\rangle$ such that $\langle x, z\rangle \in R$ and $\langle z, y\rangle \in S$ for some element z, while R^{-1}, the *converse* of R, is the set of all ordered couples $\langle x, y\rangle$ such that $\langle y, x\rangle \in R$. A

function of rank m (or a function in m variables) may be thought of as a special kind of $(m+1)$-ary relation. Thus an $(m+1)$-ary relation R is a function if, and only if, for each m-termed sequence x there exists at most one element a such that $x^\frown a \in R$.

If f is a function and A is a subset of the domain of f, then f/A will denote the function f with its domain restricted to A, i. e., f/A is the function g with domain A such that $g(x) = f(x)$ whenever $x \in A$. Given three sets A, B, and C, a function f on B^m to C, and an m-termed sequence g of functions on A^n to B, we shall understand by $f[g]$ or $f[g_0, g_1, \cdots, g_{m-1}]$ the function h on A^n to C such that

$$h(x) = f(g_0(x), g_1(x), \cdots, g_{m-1}(x)) \quad \text{whenever } x \in A^n.$$

Under the same conditions, $f\langle g\rangle$ or $f\langle g_0, g_1, \cdots, g_{m-1}\rangle$ will denote the function k on A^{mn} to C such that

$$k(x^{(0)\frown}x^{(1)\frown}\cdots{}^\frown x^{(m-1)}) = f(g_0(x^{(0)}), g_1(x^{(1)}), \cdots, g_{m-1}(x^{(m-1)}))$$
$$\text{whenever } x^{(0)}, x^{(1)}, \cdots, x^{(m-1)} \in A^n.$$

We shall refer to $f[g]$ and $f\langle g\rangle$ as the *composition* and the *superposition*, respectively, of f and g. The notion of superposition could clearly be extended to functions $g_0, g_1, \cdots$ of different ranks, but this generalization is not important for our purposes. A function f of rank m will be called an *identity function* if, for some $j < m$, we have $f(x) = x_j$ whenever x is an m-termed sequence belonging to the domain of f.

By an *algebra* we shall mean a finite, infinite, or transfinite sequence

$$\mathfrak{A} = \langle A, f_0, f_1, \cdots, f_\xi, \cdots\rangle$$

where A is a non-empty set and each f_ξ is a function of some finite rank m_ξ with domain A^{m_ξ} and counterdomain a subset of A. In this context the term 'operation' is sometimes used instead of the term 'function'; $f_0, f_1, \cdots, f_\xi, \cdots$ are referred as *fundamental operations* of $\mathfrak{A}$. By a subalgebra of $\mathfrak{A}$ we understand a sequence

$$\mathfrak{B} = \langle B, f_0/B^{m_0}, f_1/B^{m_1}, \cdots, f_\xi/B^{m_\xi}, \cdots\rangle$$

where B is a non-empty subset of A such that $f_\xi(x) \in B$ whenever $x \in B^{m_\xi}$. Sometimes we shall, however, speak of the subalgebra

$$\langle B, f_0, f_1, \cdots, f_\xi, \cdots\rangle$$

of $\mathfrak{A}$, meaning actually the algebra $\mathfrak{B}$ described above. Frequently we consider an algebra as a sequence where the first term is a non-empty set A

while the remaining terms are functions or certain distinguished elements of A. This is justified since we could replace each such element a by a function f of rank 0 such that $f(\Lambda) = a$. Thus we shall consider a Boolean algebra as a system

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1 \rangle$$

where $0, 1 \in A$, and $+$ and $\cdot$ are binary operations subject to certain familiar conditions.

We shall employ such familiar notions from general algebra as those of isomorphism ($\mathfrak{A} \cong \mathfrak{B}$), homomorphism, cardinal—or direct—product of two algebras ($\mathfrak{A} \times \mathfrak{B}$), or of arbitrarily many algebras, cardinal—or direct—power, simple algebra, and indecomposable—or directly indecomposable—algebra. Also the less familiar notions of a similarity class of algebras, an algebraic function over a similarity class, and an equationally definable class of algebras will play some part in our discussion.[3] Two algebras

$$\mathfrak{A} = \langle A, f_0, f_1, \cdots, f_\xi, \cdots \rangle \text{ with } \xi < \alpha$$

and

$$\mathfrak{B} = \langle B, g_0, g_1, \cdots, g_\xi, \cdots \rangle \text{ with } \xi < \beta$$

(where α and β are any finite or transfinite ordinals) are called *similar* if $\alpha = \beta$, and f_ξ and g_ξ are functions of the same rank for every $\xi < \alpha$. The class consisting of all algebras which are similar to a certain given algebra is referred to as a *similarity class*. Given a similarity class K, a function f is called *algebraic over* K if roughly speaking, f correlates with every algebra $\mathfrak{A}$ in K a function $f_{\mathfrak{A}}$ on A^m to A (m being some natural number) which is obtained by iterated composition from the fundamental operations of $\mathfrak{A}$. If, e. g., K is the similarity class to which the Boolean algebras belong, and if, for every algebra $\mathfrak{A} = \langle A, +, 0, \cdot, 1 \rangle$ in K, $f_{\mathfrak{A}}$ is a function on A^3 to A such that

$$f_{\mathfrak{A}}(x, y, z) = (x + y) \cdot z \text{ for any } x, y, z \in A,$$

then f is an algebraic function over K. A subclass L of K is called *equationally definable* if it can be characterized by means of a finite or infinite system of axioms all of which have the form of equations between algebraic functions over K. Speaking more precisely, L is equationally definable if there is a set S of couples $\langle f, g \rangle$ of algebraic functions over K such that an algebra $\mathfrak{A}$ in K belongs to L if, and only if, $f_{\mathfrak{A}} = g_{\mathfrak{A}}$ for every couple $\langle f, g \rangle \in S$.

[3] For these notions see McKinsey-Tarski [1] where further bibliographic references (in particular to papers of G. Birkhoff) can also be found.

Thus, for instance, Boolean algebras as they are conceived here do not constitute an equationally definable class; if, however, we agree to treat Boolean algebras as systems

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, {}^- \rangle$$

with an additional unary operation $^-$ (the operation of complementation), then the class of Boolean algebras becomes equationally definable—though, of course, within a different similarity class.

In connection with the representation theorem we shall consider systems

$$\mathfrak{A} = \langle A, R_0, R_1, \cdots, R_\xi, \cdots \rangle$$

where A is a non-empty set and each R_ξ is a subrelation of $A^{m_\xi+1}$ for some finite m_ξ. Such systems will be called *algebras in the wider sense.*

When referring to Boolean algebras, the symbols $+$, $\cdot$, $\sum$, $\prod$, $\leq$, $\geq$, 0, 1 will have their usual meaning; the complement of an element x will be denoted by x^-, and the difference of two elements x and y, i. e., $x \cdot y^-$, by $x - y$. Even when several Boolean algebras are being considered at one time these symbols will be used for all of them, it being clear from the context which algebra is referred to each time. However, in the case of two Boolean algebras where one is a subalgebra of the other, the symbols $\sum$ and $\prod$ will always refer to the larger algebra. When applied to sequences of elements of a Boolean algebra, the symbols $+$, $\cdot$, $-$, $^-$, $\sum$, $\prod$, $\leq$, and $\geq$ should be interpreted in the sense of the cardinal power of the given algebra. Thus, e. g., if x and y are two such m-termed sequences, then $x \leq y$ will mean that $x_j \leq y_j$ for every $j < m$. Similarly, if f and g are functions on an arbitrary set I to a Boolean algebra, then $f \leq g$ will mean that $f(i) \leq g(i)$ for every $i \in I$.

Section 1.

Additive Functions and Conjugate Functions.

Throughout the first part of this section we consider a fixed Boolean algebra

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1 \rangle$$

and let At denote the set consisting of 0 and all the atoms of $\mathfrak{A}$.

DEFINITION 1.1. *A function f on A^m to A is called*

(i) *normal if, given any $j < m$ and a sequence $x \in A^m$ such that $x_j = 0$, we always have $f(x) = 0$;*

(ii) *monotonic if, given two sequences* $x, y \in A^m$ *such that* $x \leq y$, *we always have* $f(x) \leq f(y)$;

(iii) *additive if, given any* $j < m$ *and two sequences* $x, y \in A^m$ *such that* $x_p = y_p$ *whenever* $j \neq p < m$, *we always have*

$$f(x + y) = f(x) + f(y);$$

(iv) *completely additive if, given any* $j < m$, *a non-empty set* I, *and sequences* $x^{(i)} \in A^m$ *with* $i \in I$ *such that* $x_p^{(i)} = x_p^{(i')}$ *whenever* $i, i' \in I$ *and* $j \neq p < m$, *and such that* $\sum_{i \in I} x^{(i)}$ *exists, then* $\sum_{i \in I} f(x^{(i)})$ *also exists and we have*

$$f(\sum_{i \in I} x^{(i)}) = \sum_{i \in I} f(x^{(i)}).$$

Obviously, every additive function is monotonic, and every completely additive function is additive.

THEOREM 1.2. *The operations of addition and multiplication in* $\mathfrak{A}$ *are completely additive functions, the latter being also normal. Furthermore, if* f *is an identity function on* A^m *to* A, *or if* $a \in A$ *and* f *is the function on* A^m *to* A *defined by the formula*

$$f(x) = a \text{ for any } x \in A^m,$$

then f *is completely additive.*

Proof. Obvious.

THEOREM 1.3. *If* f *is an additive (or completely additive) function on* A^m *to* A *and if* g *is the function on* A^{m+n} *to* A *defined by the formula*

$$g(x^\frown y) = f(x) \text{ for any } x \in A^m \text{ and } y \in A^n,$$

then g *is additive (or completely additive).*

Proof. Obvious.

THEOREM 1.4. *If* f *is an additive (or completely additive) function on* A^{m+n} *to* A *and if* $a \in A^n$ *and* g *is the function on* A^m *to* A *defined by the formula*

$$g(x) = f(x^\frown a) \text{ for any } x \in A^m,$$

then g *is additive (or completely additive).*

Proof. Obvious.

THEOREM 1.5. *A function* f *on* A *to* A *is additive (or completely*

additive) if, and only if, there exist a normal and additive (or completely additive) function g on A to A and an element $a \in A$ such that

$$f(x) = g(x) + a \text{ for any } x \in A.$$

Similarly, a function f on A^2 to A is additive (or completely additive) if, and only if, there exist a normal and additive (or completely additive) function g on A^2 to A, two normal and additive (or completely additive) function h and k on A to A, and an element $a \in A$ such that

$$f(x) = g(x) + h(x_0) + k(x_1) + a \text{ for any } x \in A^2.$$

Proof. Assume that f is an additive function on A to A. We let

$$a = f(0), \text{ and } g(x) = f(x) - a \text{ for every } x \in A.$$

By 1.1, g is normal and additive, and we have

$$f(x) = g(x) + a \text{ for every } x \in A. \tag{1}$$

(Similarly, if f is completely additive, then g is also completely additive.) Conversely, if $a \in A$, g is a normal and additive (or completely additive) function on A to A, and (1) holds, then f is additive (or completely additive) by 1.1.

To prove the second part of the theorem we proceed in an analogous manner; we let

$$a = f(0, 0),$$
$$h(x) = f(x, 0) - a \text{ and } k(x) = f(0, x) - a \text{ for every } x \in A,$$
$$g(x) = f(x) - (h(x_0) + k(x_1) + a) \text{ for every } x \in A^2.$$

Similar theorems can of course be obtained for functions of higher rank. From this we see that it would not be a serious loss of generality if we restricted our discussion to normal functions, as in fact we will do in Section 3.

THEOREM 1.6 (GENERAL DISTRIBUTIVE LAW). *Let f be a function on A^m to A. Then the following conditions are equivalent:*

(i) *The function f is additive.*

(ii) *If $K_0, K_1, \cdots, K_{m-1}$ are finite, non-empty subsets of A and if $L = K_0 \times K_1 \times \cdots \times K_{m-1}$, then*

$$f(\sum_{x \in L} x) = \sum_{x \in L} f(x).$$

Similarly, the following conditions are equivalent:

(i′) *The function f is completely additive.*

(ii′) *If $K_0, K_1, \cdots, K_{m-1}$ are non-empty subsets of A, if $L = K_0 \times K_1 \times \cdots \times K_{m-1}$, and if $\sum_{x \in L} x$ exists, then $\sum_{x \in L} f(x)$ also exists and we have*

$$f(\sum_{x \in L} x) = \sum_{x \in L} f(x).$$

Proof. By induction using 1.1 and 1.4.

THEOREM 1.7. *Suppose $\mathfrak{A}$ is atomistic and f is a function on A^m to A. Then f is completely additive if, and only if,*

$$f(x) = \sum_{x \geq u \in At^m} f(u) \text{ for any } x \in A^m.$$

Proof. The necessity of this condition follows from 1.6. Assume that the condition is satisfied. Consider a number $j < n$, a non-empty set I, and sequences $x^{(i)} \in A^m$ with $i \in I$ such that

$$x_p^{(i)} = x_p^{(i')} \text{ whenever } i, i' \in I \text{ and } j \neq p < m,$$

and such that $\sum_{i \in I} x^{(i)}$ exists. Let

$$L_i = \mathop{\mathrm{E}}_{u} [u \in At^m \text{ and } u \leq x^{(i)}] \text{ for any } i \in I, \text{ and } K = \bigcup_{i \in I} L_i.$$

Observe that

$$u \in K \text{ if, and only if, } \sum_{i \in I} x^{(i)} \geq u \in At^m.$$

Therefore

$$f(\sum_{i \in I} x^{(i)}) = \sum_{u \in K} f(u) = \sum_{i \in I} \sum_{u \in L_i} f(u) = \sum_{i \in I} f(x^{(i)}).$$

Hence f is completely additive by 1.1(iv).

THEOREM 1.8. *Suppose $\mathfrak{A}$ is atomistic. For any completely additive functions f and g on A^m to A we then have:*

(i) *$f \leq g$ if, and only if, $f(x) \leq g(x)$ for any $x \in A^m$.*

(ii) *$f = g$ if, and only if, $f(x) = g(x)$ for any $x \in A^m$.*

Proof. By 1.7.

THEOREM 1.9. *Let f be an additive (or completely additive) function on A^m to A and let $g_0, g_1, \cdots, g_{m-1}$ be additive (or completely additive) functions on A^n to A. Then $f\langle g_0, g_1, \cdots, g_{m-1}\rangle$ is an additive (or completely additive) function on A^{mn} to A.*

Proof. Assume that $f, g_0, g_1, \cdots, g_{m-1}$ are additive and let

$$h = f\langle g_0, g_1, \cdots, g_{m-1}\rangle. \tag{1}$$

Suppose $j < mn$, and consider two sequences $x, y \in A^{mn}$ such that

$$x_p = y_p \text{ whenever } j \neq p < mn. \tag{2}$$

Then there exist sequences $z^{(0)}, z^{(1)}, \cdots, z^{(m-1)}$, $u^{(0)}, u^{(1)}, \cdots, u^{(m-1)} \in A^n$ such that

$$x = z^{(0)} \frown z^{(1)} \frown \cdots \frown z^{(m-1)} \text{ and } y = u^{(0)} \frown u^{(1)} \frown \cdots \frown u^{(m-1)}. \tag{3}$$

Let q be the greatest integer such that $q \leq j/n$. Then, by (2) and (3),

$$z^{(r)} = u^{(r)} \text{ whenever } q \neq r < m, \tag{4}$$

$$z_s^{(q)} = u_s^{(q)} \text{ whenever } j - nq \neq s < n. \tag{5}$$

Hence, by 1.1(iii),

$$g_r(z^{(r)} + u^{(r)}) = g_r(z^r) + g_r(u^{(r)}) \text{ whenever } r < m. \tag{6}$$

Let the sequences $v, w \in A^m$ be defined by the formulas

$$v_r = g_r(z^{(r)}) \text{ and } w_r = g_r(u^{(r)}) \text{ whenever } r < m. \tag{7}$$

Then, by (1) and (3),

$$h(x) = f(v) \text{ and } h(y) = f(w).$$

Therefore, by (4), (7), and 1.1(iii),

$$h(x) + h(y) = f(v + w).$$

But by (1), (3), (6), (7), and 1.1(iii) we have

$$h(x + y) = f(v + w).$$

Consequently,

$$h(x + y) = h(x) + h(y). \tag{8}$$

Since (2) implies (8), we see by 1.1(iii) that h is additive. (In the case of complete additivity the proof is similar.)

It is not in general true that, under the hypothesis of the preceding theorem, the function $f[g_0, g_1, \cdots, g_{m-1}]$ is additive (or completely additive). In fact, it is precisely for this reason that we introduced the notion of the superposition of functions. We have, however, the following

THEOREM 1.10. *Let Φ be the smallest family of functions which satisfies the following conditions:*

(i) *If, for some m, f is an additive function on A^m to A, then $f \in \Phi$.*

(ii) *If, for some m and n, $f \in \Phi$ is a function on A^m to A and $g_0, g_1, \cdots, g_{m-1} \in \Phi$ are functions on A^n to A, then $f[g_0, g_1, \cdots, g_{m-1}] \in \Phi$.*

For a function f on A^n to A to belong to the family Φ it is necessary and sufficient that there exist, for some m, an additive function f' on A^m to A and m identity functions $h_0, h_1, \cdots, h_{m-1}$ on A^n to A such that

$$f = f'[h_0, h_1, \cdots, h_{m-1}].$$

Proof. Consider the family Φ' consisting of all functions f such that, for some m and n, f is of the form

$$f = f'[h_0, h_1, \cdots, h_{m-1}],$$

where f' is an additive function on A^m to A and $h_0, h_1, \cdots, h_{m-1}$ are identity functions on A^n to A. If we replace Φ by Φ', then (i) is clearly satisfied, and we shall show that (ii) is also satisfied, whence $\Phi \subseteq \Phi'$.

In fact, consider a function $f \in \Phi'$ on A^m to A and m functions $g_0, g_1, \cdots, g_{m-1} \in \Phi'$ on A^n to A. We then have, for some p,

$$f = f'[h_0, h_1, \cdots, h_{p-1}], \tag{1}$$

where f' is an additive function on A^p to A and $h_0, h_1, \cdots, h_{p-1}$ are identity functions on A^m to A. Let

$$k_j = h_j[g_0, g_1, \cdots, g_{m-1}] \text{ for every } j < p. \tag{2}$$

Then each k_j is a g_q for some $q < m$. Hence, for some r,

$$k_j = k'_j[l_0^{(j)}, l_1^{(j)}, \cdots, l_{r-1}^{(j)}] \text{ for } j < p \tag{3}$$

where the k'_j are additive functions on A^r to A while $l_0^{(j)}, l_1^{(j)}, \cdots, l_{r-1}^{(j)}$ are identity functions on A^n to A. The assumption that we have the same number r for all values of j is justified by 1.3. Let

$$g' = f'\langle k'_0, k'_1, \cdots, k'_{p-1}\rangle. \tag{4}$$

Then, by 1.9, g' is an additive function on A^{pn} to A. By (1) and (2) we have

$$f[g_0, g_1, \cdots, g_{m-1}] = f'[k_0, k_1, \cdots, k_{p-1}].$$

Hence, if we let

$$l = l^{(0)} \cap l^{(1)} \cap \cdots \cap l^{(p-1)},$$

then, by (3) and (4),

$$f[g_0, g_1, \cdots, g_{m-1}] = g'[l_0, l_1, \cdots, l_{pn-1}].$$

Thus $f[g_0, g_1, \cdots, g_{m-1}] \in \Phi'$. We conclude that $\Phi \subseteq \Phi'$. The inclusion in the opposite direction follows from 1.2. Hence $\Phi = \Phi'$, and the proof is complete.

We are now about to introduce and study the notion of conjugate functions.[4] The simple definition of conjugate functions proves to be very rich in consequences. At the first glance the new notion may apepar to be rather special; it is not involved in the formulations and proofs of the main theorems of this paper. However, our general discussion will lead to conjugate functions in quite a natural manner (compare Theorem 3.6 below); we shall also apply this notion in the discussion of cylindric algebras and relation algebras.

DEFINITION 1.11. *Let f and g be functions on A to A. We say that g is a conjugate of f if, for any $x, y \in A$, we have*

$$f(x) \cdot y = 0 \text{ if, and only if, } g(y) \cdot x = 0.$$

If, in particular, a function f is a conjugate of itself, then we call f self-conjugate.

THEOREM 1.12. *If f and g are functions on A to A and g is a conjugate of f, then f is a conjugate of g.*

Proof. Obvious.

In view of this theorem we shall usually say "f and g are conjugate" instead of "g is a conjugate of f."

THEOREM 1.13. *A function f on A to A has at most one conjugate function g, and this function (if it exists) is determined by the formula*

$$g(y) = \prod_{f(x) \cdot y = 0} x^- \text{ for any } y \in A.$$

Proof. Let g be a conjugate of f. By 1.11 we have, for any elements $x, y \in A$,

$$f(x) \cdot y = 0, \text{ if, and only if, } x \leq [g(y)]^-.$$

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[4] This notion was first discussed in Tarski [3] (though no special term to denote it was introduced there). Recently the same notion was studied (under another name) in Büchi [1], pp. 157 ff.

Thus, given an element $y \in A$, the largest element $x \in A$ for which $f(x) \cdot y = 0$ is $[g(y)]^-$. Consequently

$$[g(y)]^- = \sum_{f(x) \cdot y = 0} x \text{ for } y \in A.$$

Taking complement on both sides we obtain the desired formula for $g(y)$.

THEOREM 1.14. *A function f on A to A has a conjugate if, and only if, the following conditions are satisfied:*

(i) *f is normal,*

(ii) *f is completely additive,*

(iii) $\sum_{f(x) \leq y} x$ *exists for every element $y \in A$.*

Proof. Suppose that f has a conjugate g. Since $g(1) \cdot 0 = 0$, we have $f(0) \cdot 1 = 0$ by 1.11. Therefore $f(0) = 0$, and f is normal by 1.1(i). Let I be an arbitrary non-empty set and let the elements $x_i \in A$ with $i \in I$ be such that $\sum_{i \in I} x_i$ exists. Let

$$y = \sum_{i \in I} x_i. \tag{1}$$

Consider any element $z \in A$ such that

$$f(x_i) \leq z \text{ for every } i \in I. \tag{2}$$

For each $i \in I$ we then have $f(x_i) \cdot z^- = 0$ and hence, by 1.11,

$$g(z^-) \cdot x_i = 0 \text{ for any } i \in I.$$

By (1) this implies that $g(z^-) \cdot y = 0$, and therefore, by 1.11, that

$$f(y) \leq z. \tag{3}$$

It is easy to check that every step in the derivation of (3) from (2) is reversible and, hence, that the two formulas are equivalent for any $z \in A$. Consequently

$$f(y) = \sum_{i \in I} f(x_i). \tag{4}$$

Since (1) always implies (4), f is completely additive by 1.1(iv). An elementary transformation of the formula in 1.13 gives

$$[g(y^-)]^- = \sum_{f(x) \leq y} x \text{ for any } y \in A.$$

Therefore the right-hand member of this equation exists for every $y \in A$.

We have shown that if f has a conjugate, then conditions (i)-(iii) all hold. Conversely, assume now that (i)-(iii) are satisfied. Then the formula

(5) $$g(y) = [\sum_{f(x)\leq y^-} x]^- \text{ for any } y \in A$$

defines a function g on A to A. We can write (5) in the form

$$g(y) = \prod_{f(x)\cdot y=0} x^- \text{ for any } y \in A,$$

and it follows that

(6) $f(x)\cdot y = 0$ implies that $g(y)\cdot x = 0$ for any $x, y \in A$.

By (5) we have

$$[g(y)]^- = \sum_{f(x)\leq y^-} x \text{ for any } y \in A.$$

Hence by (i), (ii), and 1.1(i)(iv)

(7) $$f([g(y)]^-) = \sum_{f(x)\leq y^-} f(x) \leq y^- \text{ for any } y \in A.$$

If now $x, y \in A$ are such that $g(y)\cdot x = 0$, then $x \leq [g(y)]^-$. Applying first (ii) and then (7) we obtain $f(x) \leq f([g(y)]^-)$ and $f(x) \leq y^-$. Consequently,

(8) $g(y)\cdot x = 0$ implies that $f(x)\cdot y = 0$ for any $x, y \in A$.

From (6) and (8) we conclude by 1.11 that f and g are conjugate. The proof is complete.

The function f defined by the formula

$$f(x) = a\cdot x$$

is of course selfconjugate. Hence Theorem 1.14 implies as a particular case the familiar distributive law for binary multiplication under general addition (for $\cdot$ under $\sum$).

THEOREM 1.15. *If f and g are functions on A to A, then the following conditions are equivalent:*

(i) *f and g are conjugate.*

(ii) *$f(x - g(y)) \leq f(x) - y$ and $g(y - f(x)) \leq g(y) - x$ for any $x, y \in A$.*

(iii) *f and g are normal, and we have $f(x)\cdot y \leq f(x\cdot g(y))$ and $g(y)\cdot x \leq g(y\cdot f(x))$ for any $x, y \in A$.*

Proof. Suppose f and g are conjugate, and let $x, y \in A$. Then $f(x - g(y)) \leq f(x)$ by 1.1(iv) and 1.14, while by 1.11 we have $f(x - g(y))\cdot y = 0$. Hence

(1) $$f(x - g(y)) \leq f(x) - y.$$

Analogously we have

$$g(y-f(x)) \leq g(y)-x. \tag{2}$$

Assume, conversely, that (1) and (2) hold for any elements $x, y \in A$. If $f(x) \cdot y = 0$, then $y = y - f(x)$ and we have, by (2),

$$g(y) = g(y-f(x)) \leq g(y)-x \leq x^-,$$

so that $g(y) \cdot x = 0$. Similarly, by (1), $g(y) \cdot x = 0$ implies that $f(x) \cdot y = 0$. Hence f and g are conjugate functions. Thus (i) and (ii) are equivalent.

Suppose f and g are conjugate functions. Then f and g are normal by 1.14. By 1.1(iv) and 1.14 we also have, for any elements $x, y \in A$,

$$f(x) = f(x-g(y)) + f(x \cdot g(y)).$$

Since (i) implies (ii), we infer that

$$f(x) \leq (f(x)-y) + f(x \cdot g(y)).$$

Consequently

$$f(x) \cdot y \leq (f(x)-y) \cdot y + f(x \cdot g(y)) \cdot y \leq f(x \cdot g(y)).$$

Similarly,

$$g(y) \cdot x \leq g(y \cdot f(x)).$$

Conversely, assume that (iii) holds. If $x, y \in A$ are such that $f(x) \cdot y = 0$, then

$$g(y) \cdot x \leq g(y \cdot f(x)) = g(0) = 0$$

Similarly, $g(y) \cdot x = 0$ implies that $f(x) \cdot y = 0$. Hence f and g are conjugate by 1.11 and the proof is complete.

The next three theorems are of a rather special nature, but they will have added significance in connection with Theorems 3.5 and 3.7.

THEOREM 1.16. *If f is a function on A to A, then the following conditions are equivalent:*

(i) *f is selfconjugate.*

(ii) *f is additive, and $f([f(x)]^-) \cdot x = 0$ for any $x \in A$.*

(iii) *f is normal, and $f(x) \cdot y \leq f(x \cdot f(y))$ for any $x, y \in A$.*

Proof. If (ii) is satisfied, then by 1.1(iii).

$$f(0) \leq f([f(1)]^-) \cdot 1 = 0.$$

Hence $f(0) = 0$, and f is normal by 1.1(i). Furthermore, if $x, y \in A$, then 1.1(iii) gives with the aid of (ii)

$$f(x)\cdot y = f(x\cdot f(y))\cdot y + f(x\cdot[f(y)]^-)\cdot y$$
$$\leq f(x\cdot f(y)) + f([f(y)]^-)\cdot y = f(x\cdot f(y)).$$

Thus, (ii) implies (iii). That (iii) implies (i) and that (i) implies (ii) are immediate consequences of 1.14 and 1.15.

THEOREM 1.17. *If f is a function on A to A, then the following conditions are equivalent:*

(i) *f is additive, and $f(f(x)) = x\cdot f(1)$ for any $x \in A$.*

(ii) *$f(x)\cdot y = f(x\cdot f(y))$ for any $x, y \in A$.*

Proof. Suppose (i) holds. Then, by 1.1(iii),

$$f(0) \leq f(f(0)) = 0.$$

Hence f is normal by 1.1(i). We shall next show that

(1) $\quad x\cdot y = 0$ implies that $f(x)\cdot f(y) = 0$ for any $x, y \in A$.

In fact, if $x\cdot y = 0$, then by (i) and 1.1(iii)

$$f(f(x)\cdot f(y)) \leq f(f(x))\cdot f(f(y)) = x\cdot y\cdot f(1) = 0,$$

and consequently

$$f(x)\cdot f(y) = f(x)\cdot f(y)\cdot f(1) = f(f(f(x)\cdot f(y))) = f(0) = 0.$$

Thus (1) holds. For any elements $x, y \in A$ we have, by (i) and 1.1(iii),

$$f(x)\cdot y = f(x)\cdot y\cdot f(1) = f(x)\cdot f(f(y))$$
$$= f(x)\cdot f(x\cdot f(y)) + f(x)\cdot f(x^-\cdot f(y)).$$

Hence, by (1) and 1.1(iii),

$$f(x)\cdot y = f(x\cdot f(y)) \text{ for any } x, y \in A,$$

and (ii) is satisfied. Conversely, if (ii) holds, then

$$f(0) = f(0\cdot f(0)) = f(0)\cdot 0 = 0,$$

so that f is normal by 1.1(i). Hence the first part of (i) follows from 1.16. The second part of (i) is but a special case of (ii).

THEOREM 1.18. *If f is a function on A to A, then the following conditions are equivalent:*

(i) *f is additive and $f(f(x)) = x$ for any $x \in A$.*

(ii) *$f(1) = 1$ and $f(x)\cdot y = f(x\cdot f(y))$ for any $x \in A$.*

Proof. Suppose (i) holds. Then $f(f(1)) = 1$ and hence, by 1.1(iii), $f(1) = 1$. Therefore, by 1.17, (ii) is satisfied. The converse is an immediate consequence of 1.17.

The representation theorem and the extension theorem for Boolean algebras were mentioned in the introduction, but for our later purposes the form in which they were given there is not strong enough. As regards the representation theorem, the stronger form which we have in mind is usually stated as follows:

Every Boolean algebra is isomorphic to a set-field consisting of all open and closed sets in a totally-disconnected compact space.[5]

Before stating the corresponding form of the extension theorem, we shall introduce the notions of a regular subalgebra and a perfect extension. The results contained in the last part of this section are substantially known, and the proofs will therefore be omitted.

DEFINITION 1.19. *Let*

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1\rangle \text{ and } \mathfrak{B} = \langle B, +, 0, \cdot, 1\rangle$$

be two Boolean algebras. We say that $\mathfrak{B}$ is a regular subalgebra of $\mathfrak{A}$ and that $\mathfrak{A}$ is a perfect extension of $\mathfrak{B}$ if the following conditions are satisfied:

(i) *$\mathfrak{A}$ is complete and atomistic, and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$.*

(ii) *If I is an arbitrary set, and if the elements $x_i \in B$ with $i \in I$ are such that*

$$\sum_{i \in I} x_i = 1,$$

then there exists a finite subset J of I such that

$$\sum_{i \in J} x_i = 1.$$

(iii) *If u and v are distinct atoms of $\mathfrak{A}$, then there exists an element $b \in B$ such that*

$$u \leq b \text{ and } v \cdot b = 0.$$

DEFINITION 1.20. *Let*

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1\rangle$$

be a complete atomistic Boolean algebra, and let

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1\rangle$$

---

[5] Cf. Stone [1], Theorem 1. See the same work, Theorem 4, in connection with our Theorem 1.23.

be a regular subalgebra of $\mathfrak{A}$. *An element* $x \in A$ *is said to be*

(i) *open if* $x = \sum_{x \geq y \in B} y$;

(ii) *closed if* $x = \prod_{x \leq y \in B} y$.

THEOREM 1.21. *Let*

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1 \rangle$$

be a complete and atomistic Boolean algebra, and let

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1 \rangle$$

be a regular subalgebra of $\mathfrak{A}$. *We then have:*

(i) *For any* $x \in A$, x *is open if, and only if,* x^- *is closed.*

(ii) *For any* $x \in A$, x *is open and closed if, and only if,* $x \in B$.

(iii) *If* $x \in A$ *is closed,* I *is an arbitrary set, the elements* $y_i \in A$ *with* $i \in I$ *are open, and*

$$x \leq \sum_{i \in I} y_i,$$

then there exists a finite subset J *of* I *such that*

$$x \leq \sum_{i \in J} y_i.$$

(iv) *If* $x \in A$ *is open,* I *is an arbitrary set, the elements* $y_i \in A$ *with* $i \in I$ *are closed, and*

$$x \geq \prod_{i \in I} y_i,$$

then there exists a finite subset J *of* I *such that*

$$x \geq \prod_{i \in J} y_i.$$

(v) *If* u *is an atom of* $\mathfrak{A}$, *then* u *is closed.*

(vi) *If* u *is an atom of* $\mathfrak{B}$, *then* u *is an atom of* $\mathfrak{A}$.

The extension theorem for Boolean algebras can now be stated as follows:

THEOREM 1.22. *For any Boolean algebra* $\mathfrak{B}$ *there exists a complete and atomistic Boolean algebra* $\mathfrak{A}$ *which is a perfect extension of* $\mathfrak{B}$.

That the perfect extension $\mathfrak{A}$ is essentially determined by the Boolean algebra $\mathfrak{B}$ is shown in the next two theorems.

THEOREM 1.23. *Let*

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1 \rangle \text{ and } \mathfrak{A}' = \langle A', +, 0, \cdot, 1 \rangle$$

be two complete and atomistic Boolean algebras, and let

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1\rangle \text{ and } \mathfrak{B}' = \langle B', +, 0, \cdot, 1\rangle$$

be regular subalgebras of $\mathfrak{A}$ *and* $\mathfrak{A}'$ *respectively. If a function* ϕ *maps* $\mathfrak{B}$ *isomorphically onto* $\mathfrak{B}'$, *then there exists a unique function* ψ *which maps* $\mathfrak{A}$ *isomorphically onto* $\mathfrak{A}'$ *in such a way that* $\psi/B = \phi$.

THEOREM 1.24. *If*

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1\rangle$$

is a Boolean algebra, and if $\mathfrak{A}$ *and* $\mathfrak{A}'$ *are two perfect extensions of* $\mathfrak{B}$, *then there exists a unique function* ψ *which maps* $\mathfrak{A}$ *isomorphically onto* $\mathfrak{A}'$ *in such a way that* $\psi(x) = x$ *for every* $x \in B$.

In the following we shall use various definitions and theorems from this section without always explicitly referring to them.

Section 2.

The Extension Theorem.

Throughout the first part of this section we shall consider a fixed Boolean algebra

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1\rangle$$

which is complete and atomistic, and a regular subalgebra

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1\rangle$$

of $\mathfrak{A}$. The set consisting of 0 and all the atoms of $\mathfrak{A}$ will be denoted by At, the set of all closed elements by C. Although we associate with every function f on B^m to B a function f^+ on A^m to A (see Definition 2.1), we shall be mostly concerned with those functions f which are additive or obtainable from additive functions by means of composition. In this case f^+ is an extension of f. It is shown that if f is additive, then f^+ is completely additive. We shall also prove that if an equation involving additive functions on B^m to B is identically satisfied, then the corresponding equation involving their extensions will also hold identically. In the last part of the section these results are applied to Boolean algebras with operators.

DEFINITION 2.1. *For any function f on B^m to B, f^+ is the function on A^m to A defined by the formula*

$$f^+(x) = \sum_{x \geq y \in C^m} \prod_{y \leq z \in B^m} f(z) \text{ for any } x \in A^m.$$

THEOREM 2.2. *If f is a function on B^m to B, then*

$$f^+(y) = \prod_{y \leq z \in B^m} f(z) \text{ for any } y \in C^m.$$

Proof. By 2.1.

THEOREM 2.3. *If f is an additive (or, more generally, a monotonic) function on B^m to B, then*

$$f^+/B^m = f.$$

Proof. By 2.2 with the aid of 1.21(ii).

THEOREM 2.4. *If f is an additive function on B^m to B, then f^+ is a completely additive function on A^m to A.*

Proof. Let g be the function on A^m to A such that

$$g(x) = \sum_{x \geq u \in At^m} f^+(u) \text{ for any } x \in A^m. \tag{1}$$

By 1.7, g is completely additive. In order to prove that $f^+ = g$, we shall first show that

$$f^+(y) \leq g(y) \text{ for any } y \in C^m. \tag{2}$$

Consider a fixed sequence $y \in C^m$ and an atom v of $\mathfrak{A}$ such that

$$v \cdot g(y) = 0. \tag{3}$$

We shall show that there exists a sequence z such that

$$y \leq z \in B^m \text{ and } v \cdot f(z) = 0. \tag{4}$$

Let

$$K = \mathop{\mathrm{E}}_{u}[y \geq u \in At^m]. \tag{5}$$

For any sequence $u \in K$ we then have, by (1) and (3), $v \cdot f^+(u) = 0$. Hence, by 2.2, we can correlate with each $u \in K$ a sequence u' such that

$$u \leq u' \in B^m \text{ and } v \cdot f(u') = 0 \text{ whenever } u \in K. \tag{6}$$

We shall correlate with each sequence $u \in K$ elements $\phi_{j\,p}(u) \in B$ with $j < m$ and $p \leq m$ such that the following conditions are satisfied:

$$y_j \leq \phi_{j\,p}(u) \text{ whenever } j < p \leq m \text{ and } u \in K. \tag{7}$$

(8) $u_j \leq \phi_{j\,p}(u)$ whenever $p \leq j < m$ and $u \in K$.

(9) $v \cdot f(\phi_{0\,p}(w), \phi_{1\,p}(u), \cdots, \phi_{m-1\,p}(u)) = 0$ whenever $p \leq m$ and $u \in K$.

The elements $\phi_{j\,p}(u)$ will be defined recursively. We let

$$\phi_{j\,0}(u) = u'_j \text{ for any } j < m \text{ and } u \in K.$$

Assume that $p < m$, that $\phi_{j\,p}(u)$ has been defined for any $j < m$ and $u \in K$, and that (7), (8), and (9) are satisfied for this fixed value of p. Consider a fixed sequence $u \in K$ and the class L of all sequences $w \in K$ such that $w_j = u_j$ whenever $j < m$ and $j \neq p$. For each atom $a \leq y_p$ we have a sequence $w \in L$ such that $w_p = a$ and hence $a \leq \phi_{p\,p}(w)$. Consequently,

$$y_p \leq \sum_{w \in L} \phi_{p\,p}(w).$$

Since y_p is closed and $\phi_{p\,p}(w) \in B$, there exists, by 1.21(ii),(iii), a finite subset M of L such that

$$y_p \leq \sum_{w \in M} \phi_{p\,p}(w).$$

Let

$$\phi_{p\,p+1}(u) = \sum_{w \in M} \phi_{p\,p}(w),$$

$$\phi_{j\,p+1}(u) = \prod_{w \in M} \phi_{j\,p}(w) \text{ whenever } j < m \text{ and } j \neq p.$$

It is easy to see that (7), (8), and (9) are satisfied with p replaced by $p+1$. We now take any sequence $u \in K$, define the m-termed sequence $z \in B^m$ by the formula

$$z_j = \phi_{j\,m}(u) \text{ for } j < m,$$

and use (7) and (9) to check that (4) is satisfied. Hence, by 2.2, $v \cdot f^+(y) = 0$. Since this is true whenever $y \in C^m$, v is an atom of $\mathfrak{A}$, and (3) is satisfied, we conclude that (2) holds.

By (1), (2), 2.1, and 2.2 we have, for any $x \in A^m$,

$$f^+(x) = \sum_{x \geq y \in C^m} f^+(y) \leq \sum_{x \geq y \in C^m} g(y) \leq g(x).$$

Thus $f^+ \leqq g$. The opposite inequality follows immediately from (1) and 2.1. Hence $f^+ = g$, and the proof is complete.

THEOREM 2.5. *If f is an additive function on B^m to B and g is a completely additive function on A^m to A such that $g/B^m \leq f$, then $g \leq f^+$.*

Proof. For any $u \in At^m$ we have, by 2.2 and the hypothesis,

$$f^+(u) = \prod_{u \leq y \in B^m} f(y) \geq \prod_{u \leq y \in B^m} g(y) \geq g(u).$$

Hence the conclusion follows by 1.8(i), 2.4, and the hypothesis.

It may be interesting to notice that, in general, if f is an additive function on B^m to B, then there exist, besides f^+, other completely additive functions g on A^m to A which satisfy the condition $g/B^m = f$. To give a simple example, let f be the function on B to B such that

$$f(0) = 0, \text{ and } f(x) = 1 \text{ whenever } 0 \neq x \in B.$$

It follows immediately from 2.1 that

$$f^+(0) = 0, \text{ and } f^+(x) = 1 \text{ whenever } 0 \neq x \in A.$$

Observe that, if B is infinite, then At is not a subset of B. For, assume that $At \subseteq B$. Since

$$\sum_{u \in At} u = 1,$$

there exists, by 1.19(ii), a finite subset K of At such that

$$\sum_{u \in K} u = 1.$$

But this clearly implies that K contains all the atoms of $\mathfrak{A}$. Hence At is finite, which in turn implies that A and B are finite. Assuming that B is infinite, we can therefore choose an atom u of $\mathfrak{A}$ such that $u \notin B$. Let the function g on A to A be defined by the formulas

$$g(0) = 0 = g(u), \text{ and } g(x) = 1 \text{ whenever } x \in A \text{ and } x \neq 0, u.$$

Obviously, g is completely additive, and we have $g/B = f$ while $g \neq f^+$.

If f is an additive function on B^m to B, and if g is a completely additive function on A^m to A such that $g/B^m = f$, then $g \leq f^+$ by 2.5. We may express this by saying that f^+ is the largest possible extension of f which is completely additive. However, if we drop the condition that the extension is completely additive and consider instead all additive extensions of f, then f^+ no longer has this property. In fact, the largest possible extension of f which is additive is the function g such that

$$g(x) = \prod_{x \leq y \in B^m} f(y) \text{ for any } x \in A^m,$$

and it is easy to show by examples that we do not in general have $g = f^+$. As a matter of fact, if B is infinite, and if f is an identity function on B^m to B, then $f^+ \neq g$. For, as we will prove in the next theorem, f^+ is then the corresponding identity function on A^m to A. Hence $f^+ = g$ implies that every element of A is closed and, therefore, by 1.21(i), (ii), that $A = B$. But we have already shown that this is true only when B is finite.

THEOREM 2.6. *If f and g are the functions on B^2 to B defined by the formulas*

$$f(x) = x_0 + x_1 \text{ and } g(x) = x_0 \cdot x_1 \text{ for any } x \in B^2,$$

then

$$f^+(x) = x_0 + x_1 \text{ and } g^+(x) = x_0 \cdot x_1 \text{ for any } x \in A^2.$$

Furthermore, if $j < m$ and $b \in B$, and if h and k are the functions on B^m to B defined by the formulas

$$h(x) = b \text{ and } k(x) = x_j \text{ for any } x \in B^m,$$

then

$$h^+(x) = b \text{ and } k^+(x) = x_j \text{ for any } x \in A^m.$$

Proof. For any $u \in At^2$ we have, by 2.2,

$$f^+(u) = \prod_{u \leq y \in B^2} (y_0 + y_1). \tag{1}$$

Therefore

$$u_0 + u_1 \leq f^+(u). \tag{2}$$

Consider any atom a of $\mathfrak{A}$ with $a \cdot u_0 = 0 = a \cdot u_1$. By 1.19(iii) there exists a sequence $y \in B^2$ such that $u \leq y$ and $a \cdot y_0 = 0 = a \cdot y_1$. It follows by (1) that $a \cdot f^+(u) = 0$ for every such atom a. Consequently $f^+(u) \leq u_0 + u_1$. Together with (2) this gives

$$f^+(u) = u_0 + u_1.$$

Since this formula holds whenever $u \in At^2$, we conclude by 1.8(ii) and 2.4 that

$$f^+(x) = x_0 + x_1 \text{ for any } x \in A^2.$$

To show that

$$g^+(x) = x_0 \cdot x_1 \text{ for any } x \in A^2,$$

we proceed in an analogous manner.

By 1.20(ii), 1.21(v) and 2.2 we have $k^+(u) = u_j$ for any $u \in At^m$. Hence, by 1.8(ii) and 2.4,

$$k^+(x) = x_j \text{ for any } x \in A^m.$$

By 2.1 we have

$$h^+(x) = b \text{ for any } x \in A^m.$$

The proof is complete.

THEOREM 2.7. *If f is a function on B^m to B and g is the function on B^{m+n} to B such that*

$$g(x^\frown y) = f(x) \text{ for any } x \in B^m \text{ and } y \in B^n,$$

then

$$g^+(x^\frown y) = f^+(x) \text{ for any } x \in A^m \text{ and } y \in A^n.$$

Proof. By 2.1.

THEOREM 2.8. *Let f be an additive function on B^m to B, and let $g_0, g_1, \cdots, g_{m-1}$ be additive functions on B^n to B. Then*

$$(f\langle g_0, g_1, \cdots, g_{m-1}\rangle)^+ = f^+\langle g^+_0, g^+_1, \cdots, g^+_{m-1}\rangle.$$

Proof. Let

$$(1) \qquad h = f\langle g_0, g_1, \cdots, g_{m-1}\rangle, \qquad k = f^+\langle g^+_0, g^+_1, \cdots, g^+_{m-1}\rangle.$$

Then h^+ and k are completely additive functions on A^{mn} to A by 1.9 and 2.4. In order to prove that $h^+ = k$, it is therefore sufficient to show that

$$(2) \qquad h^+(y) = k(y) \text{ for any } y \in C^{mn}.$$

For each sequence $y \in C^{mn}$ let y' be the m-termed sequence defined by the formula

$$(3) \qquad y'_j = g^+_j(y_{j\,n}, y_{j\,n+1}, \cdots, y_{j\,n+n-1}) \text{ for any } j < m.$$

If $y \in C^{mn}$, then $y' \in C^m$ by (3) and 2.2. Hence, by (1) and 2.2,

$$(4) \quad h^+(y) = \prod_{y \le z \in B^{mn}} f(z') \text{ and } k(y) = \prod_{y' \le t \in B^m} f(t) \text{ for any } y \in C^{mn}.$$

By (3), 2.3, and the additivity of $g_0, g_1, \cdots, g_{m-1}$ we have

$$y' \leqq z' \in B^m \text{ whenever } y \in C^{mn},\ z \in B^{mn}, \text{ and } y \le z.$$

Consequently, by (4) and the additivity of f,

$$(5) \qquad k(y) \le h^+(y) \text{ for any } y \in C^{mn}.$$

To prove the inequality in the opposite direction, it is clearly sufficient to show that if

$$(6) \qquad y \in C^{mn} \text{ and } y' \le t \in B^m,$$

then there exists a sequence z such that

$$(7) \qquad y \le z \in B^{mn} \text{ and } z' \le t.$$

Assume that (6) holds and let

$$u^{(j)} = y_{j\,n}{}^\frown y_{j\,n+1}{}^\frown \cdots {}^\frown y_{j\,n+n-1} \text{ for any } j < m.$$

Then

$$\prod_{u^{(j)} \le v \in B^n} g_j(v) = y'_j \le t_j \text{ for any } j < m.$$

Since $t_j \in B$ by (6), and $g_j(v) \in B$ for $v \in B^n$, it follows by 1.21(ii)(iv) that there exists, for each $j < m$, a finite set K_j of sequences v with $u^{(j)} \leq v \in B^n$ such that

$$\prod_{v \in K_j} g_j(v) \leq t_j \text{ for any } j < m.$$

Let z be the mn-termed sequence defined by the formula

$$z_{jn+p} = \prod_{v \in K_j} v_p \text{ for any } j < m \text{ and } p < n.$$

It is easy to check that (7) is satisfied. Thus (6) always implies (7), and we conclude that the inclusion in (5) can be replaced by an equality. The proof is complete.

THEOREM 2.9. *Let f be a monotonic function on B^m to B, and let $g_0, g_1, \cdots, g_{m-1}$ be identity functions on B^n to B. Then*

$$(f[g_0, g_1, \cdots, g_{m-1}])^+ = f^+[g^+_0, g^+_1, \cdots, g^+_{m-1}].$$

Proof. Let

$$h = f[g_0, g_1, \cdots, g_{m-1}]. \tag{1}$$

For each $x \in A^n$, let x' be the m-termed sequence such that

$$x'_j = g^+_j(x) \text{ for any } j < m. \tag{2}$$

We shall prove the following two statements which will be needed later:

(3) If $x \in A^n$ and $x' \geq z \in C^m$, then $x \geq y$ and $y' \geq z$ for some $y \in C^n$.

(4) If $x \in A^n$ and $x' \leq t \in B^m$, then $x \leq s$ and $s' \leq t$ for some $s \in B^n$.

For each $p \leq n$, let k_p be the function on B^n to B such that

$$k_p(x) = x_p \text{ for any } x \in B^n, \tag{5}$$

and let

$$K_p = \mathop{\mathrm{E}}_j[j < m \text{ and } g_j = k_p]. \tag{6}$$

The sequences y and s are defined by the formulas

$$y_p = \sum_{j \in K_p} z_j \text{ and } s_p = \prod_{j \in K_p} t_j \text{ for any } p < n. \tag{7}$$

From the finiteness of the sets K_p it follows that $y \in C^n$ and $t \in B^n$. If $j \in K_p$, then, by (2), (5), and (6),

$$x'_j = g^+_j(x) = k^+_p(x) = x_p.$$

Consequently, by our assumptions regarding z and t,

$$z_j \leq x_p \leq t_j \text{ whenever } p < n \text{ and } j \in K_p.$$

Therefore, by (7),

$$y_p \leq x_p \leq s_p \text{ for any } p < n. \tag{8}$$

If $j < m$, then $j \in K_p$ for some $p < n$. Hence, by (2), (5), (6), and (7),

$$y'_j = y_p \geq z_j \text{ and } s'_j = s_p \leq t_j.$$

We therefore have

$$y' \geq z \text{ and } s' \leq t.$$

Together with (8) this shows that the conclusions of (3) and (4) are satisfied.

Consider now a fixed sequence $x \in A^n$. By (1), (2), and 2.1 we have

$$h^+(x) = \sum_{x \geq y \in C^n} \prod_{y \leq s \in B^n} f(s'), \tag{9}$$

$$f^+(x') = \sum_{x' \geq z \in C^m} \prod_{z \leq t \in B^m} f(t). \tag{10}$$

For any sequence y with $x \geq y \in C^n$, we have, by (2), 2.1, and 2.2, $x' \geq y' \in C^m$. Furthermore, if $y' \leq t \in B^m$, then there exists, by (4), a sequence $s \in B^n$ such that $y \leq s$, $s' \leq t$, and hence $f(s') \leq f(t)$. We thus have

$$\prod_{y \leq s \in B^n} f(s') \leq \prod_{y' \leq t \in B^m} f(t).$$

Since this is true whenever $x \geq y \in C^n$, we conclude, by (9) and (10), that

$$h^+(x) \leq f^+(x'). \tag{11}$$

For any sequence z with $x' \geq z \in C^m$ there exists, by (3), a sequence $y \in C^n$ such that $x \geq y$, $y' \geq z$, and hence

$$\prod_{z \leq t \in B^m} f(t) \leq \prod_{y' \leq t \in B^m} f(t).$$

Furthermore, since $y \leq s \in B^n$ always implies that $y' \leq s' \in B^m$, we have

$$\prod_{y' \leq t \in B^m} f(t) \leq \prod_{y \leq s \in B^n} f(s').$$

For each sequence z with $x' \geq z \in C^m$ there exists, therefore, a sequence y such that $x \geq y \in C^n$ and

$$\prod_{z \leq t \in B^m} f(t) \leq \prod_{y \leq s \in B^n} f(s').$$

Consequently, by (9) and (10),

$$f^+(x') \leq k^+(x). \tag{12}$$

Since (11) and (12) hold for any $x \in A^n$, the conclusion of the theorem follows from (1) and (2).

THEOREM 2.10. *Let Φ be the smallest family of functions which satisfies the following conditions:*

(i) *If, for some m, f is an additive function on B^m to B, then $f \in \Phi$.*

(ii) *If, for some m and n, $f \in \Phi$ is a function on B^m to B while $g_0, g_1, \cdots, g_{m-1} \in \Phi$ are functions on B^n to B, then $f[g_0, g_1, \cdots, g_{m-1}] \in \Phi$.*

Given any function $f \in \Phi$ on B^m to B and m functions $g_0, g_1, \cdots, g_{m-1} \in \Phi$ on B^n to B, we then have

$$(f[g_0, g_1, \cdots, g_{m-1}])^+ = f^+[g^+_0, g^+_1, \cdots, g^+_{m-1}].$$

Proof. By 1.10 there exist, for some p, an additive function f' on B^p to B, and identity functions $h_0, h_1, \cdots, h_{p-1}$ on B^m to B such that

$$f = f'[h_0, h_1, \cdots, h_{p-1}]. \tag{1}$$

Let

$$k_j = h_j[g_0, g_1, \cdots, g_{m-1}] \text{ for any } j < p. \tag{2}$$

Then each k_j is one of the functions $g_0, g_1, \cdots, g_{m-1}$, whence $k_j \in \Phi$. We can therefore represent k_j in the form

$$k_j = k'_j[l_0^{(j)}, l_1^{(j)}, \cdots, l_{q-1}^{(j)}] \tag{3}$$

where k'_j is an additive function on B^q to B, and $l_0^{(j)}, l_1^{(j)}, \cdots, l_{q-1}^{(j)}$ are identity functions on B^n to B. The assumption that we have the same number q for all values of j is justified by 1.3. By (1) and (2) we have

$$f[g_0, g_1, \cdots, g_{m-1}] = f'[k_0, k_1, \cdots, k_{p-1}]. \tag{4}$$

Let

$$g' = f'\langle k'_0, k'_1, \cdots, k'_{p-1}\rangle \text{ and } l = l^{(0)} \cap l^{(1)} \cap \cdots \cap l^{(p-1)}. \tag{5}$$

Then, by (3) and (4),

$$f[g_0, g_1, \cdots, g_{m-1}] = g'[l_0, l_1, \cdots, l_{mp-1}].$$

Consequently, by 2.9,

$$(6) \qquad (f[g_0, g_1, \cdots, g_{m-1}])^+ = g'^+[l^+_0, l^+_1, \cdots, l^+_{mp-1}].$$

By (1) and 2.9 we have

$$f^+ = f'^+[h^+_0, h^+_1, \cdots, h^+_{p-1}]$$

while, by 2.6,

$$k^+_j = h^+_j[g^+_0, g^+_1, \cdots, g^+_{m-1}] \text{ for any } j < p.$$

Consequently

$$(7) \qquad f^+[g^+_0, g^+_1, \cdots, g^+_{m-1}] = f'^+[k^+_0, k^+_1, \cdots, k^+_{p-1}].$$

From (3) and 2.9 it follows that

$$(8) \qquad k^+_j = k'^+_j[l_0^{(j)+}, l_1^{(j)+}, \cdots, l_{q-1}^{(j)+}] \text{ for any } j < p$$

while, by (5) and 2.8,

$$g'^+ = f'^+\langle k'^+_0, k'^+_1, \cdots, k'^+_{p-1}\rangle.$$

Hence, by (5), (7), and (8),

$$(9) \qquad f^+[g^+_0, g^+_1, \cdots, g^+_{m-1}] = g'^+[l^+_0, l^+_1, \cdots, l^+_{mp-1}].$$

The conclusion follows from (6) and (9).

It follows from the preceding theorem that if an equation involving additive functions $f_0, f_1, \cdots, f_{m-1}$ on B^n to B holds for every sequence $x \in B^n$, then the corresponding equation with the functions $f^+_0, f^+_1, \cdots, f^+_{m-1}$ holds for every sequence $x \in A^n$. It would be natural to ask at this point what other properties are preserved by the extension. We have not made a detailed study of this question, and shall only make a few remarks here.

We first notice that a similar result can be obtained for certain implications between two equations, as well as between an inequality and an equation. In fact, we have

THEOREM 2.11. *Let Φ be as in Theorem 2.10, and let $f, g, h \in \Phi$ be functions on B^m to B.*

(i) *If*

$$f(x) = 0 \text{ implies that } g(x) = h(x) \text{ for every } x \in B^m,$$

then

$$f^+(x) = 0 \text{ implies that } g^+(x) = h^+(x) \text{ for every } x \in A^m.$$

(ii) *If*

$$f(x) \neq 0 \text{ implies that } g(x) = h(x) \text{ for every } x \in B^m,$$

then

$$f^+(x) \neq 0 \text{ implies that } g^+(x) = h^+(x) \text{ for every } x \in A^m.$$

Proof. Let k be the function on B to B such that

$$k(0) = 0, \text{ and } k(x) = 1 \text{ whenever } 0 \neq x \in B.$$

By the hypothesis of (i) we then have

$$(1) \qquad k(f(x)) + g(x) = k(f(x)) + h(x) \text{ for every } x \in B^m.$$

Consequently, the function k being additive, we get by 2.6 and 2.10

$$(2) \qquad k^+(f^+(x)) + g^+(x) = k^+(f^+(x)) + h^+(x) \text{ for every } x \in A^m.$$

By 2.1 and the definition of k we have

$$k^+(0) = 0, \text{ and } k^+(x) = 1 \text{ whenever } 0 \neq x \in A.$$

Hence, by (2), we obtain the conclusion of (i):

$$f^+(x) = 0 \text{ implies that } g^+(x) = h^+(x) \text{ for every } x \in A^m.$$

The proof of (ii) is analogous; we change $+$ to $\cdot$ in (1) and (2).

As a corollary of the preceding theorem we obtain

THEOREM 2.12. *Let f and g be functions on B to B. If f and g are conjugate, then f^+ and g^+ are conjugate.*

Proof. By 1.11, 1.14, and 2.11.

Theorem 2.11 can be generalized by considering implications in which the hypothesis is, not an equation of the form $f(x) = 0$ or an inequality of the form $f(x) \neq 0$, but a conjunction or disjunction of formulas of both these types. To obtain this generalization, we argue as in the proof of 2.11 and, by applying the function k defined there, we show that every implication of the kind considered can be equivalently replaced by a single equation or a conjunction of equations. For instance, the implication

$$\text{if } f_0(x) = 0 \text{ and } f_1(x) \neq 0, \text{ then } g(x) = h(x)$$

proves to be equivalent to the equation

$$k(f_0(x)) + k(f_1(x)) \cdot g(x) = k(f_0(x)) + k(f_1(x)) \cdot h(x).$$

On the other hand, we can give an example of a simple formula which is identically satisfied by an additive function f in a certain Boolean algebra $\mathfrak{B} = \langle B, +, 0, \cdot, 1\rangle$, but which is not satisfied by any additive function g (whether an extension of f or not) in $\mathfrak{A} = \langle A, +, 0, \cdot, 1\rangle$; $\mathfrak{A}$ is assumed to be, not necessarily a perfect extension of $\mathfrak{B}$, but an arbitrary complete Boolean algebra of which $\mathfrak{B}$ is a subalgebra. In fact, let U be the set of all positive integers, and let B be the family consisting of all finite subsets of U and of their complements (with respect to U). Suppose that $+$ and $\cdot$ are set-addition and set-multiplication, 0 is the empty set, and 1 is the set U. For every x in B, let $f(x)$ be the set consisting of the integer 2 and of all integers $n + 2$ where n is in x. As is easily seen, f is an additive function on B to B satisfying the condition:

$$f(x) \neq x \text{ for every } x \in B.$$

This condition implicitly involves, in addition to f, another additive function, namely the identity function on B to B, but it implies the following condition which involves only f:

$$f(f(x)) \neq f(x) \text{ for every } x \in B.$$

On the other hand there is no additive function g in $\mathfrak{A}$ which satisfies the corresponding condition

$$g(g(x)) \neq g(x) \text{ for every } x \in A.$$

For, g being any additive (or, more generally, monotonic) function on A to A, we obtain an element $c \in A$ for which $g(c) = c$ and hence $g(g(c)) = g(c)$ by putting

$$c = \sum_{x \leq g(x)} x.^{6}$$

One might ask whether Theorem 2.10 could be generalized by replacing Φ by some larger family of functions; in particular, it would be natural to consider the family of all monotonic functions. However, it can be shown that there exist monotonic functions f and g in one variable such that $(f[g])^+ \neq f^+[g^+]$. In fact let

[6] Cf. Tarski [1].

$$\mathfrak{D} = \langle D, +, 0, \cdot, 1\rangle \text{ and } \mathfrak{E} = \langle E, +, 0, \cdot, 1\rangle$$

be Boolean algebras such that $\mathfrak{D}$ is infinite and complete, and $\mathfrak{E}$ is a perfect extension of $\mathfrak{D}$. Let U be the set of all the atoms of $\mathfrak{E}$, let B be the set of all functions on U to D, and let θ and I be the functions on U to D such that

$$\Theta(u) = 0 \text{ and } \mathrm{I}(u) = 1 \text{ for any } u \in U.$$

Suppose

$$\mathfrak{A} = \langle A, +, \Theta, \cdot, \mathrm{I}\rangle$$

is a perfect extension of the Boolean algebra

$$\mathfrak{B} = \langle B, +, \Theta, \cdot, \mathrm{I}\rangle.$$

We now take a fixed element $w \in U$ and define the functions f and g on B to B by the formulas

$$f(\alpha) = \Theta \text{ whenever } \alpha \in B \text{ and } \alpha(w) \neq 1,$$

$$f(\alpha) = \mathrm{I} \text{ whenever } \alpha \in B \text{ and } \alpha(w) = 1,$$

$$g(\alpha)(w) = \sum_{u \in U}' \alpha(u), \text{ and } g(\alpha)(v) = 0 \text{ whenever } \alpha \in B \text{ and } w \neq v \in U.$$

Here $\sum'$ refers to the sum in the Boolean algebra $\mathfrak{D}$. Clearly g is additive and f is monotonic. It will be shown that $(f[g])^+ \neq f^+[g^+]$.

For each $u \in U$ let

$$K_u = \mathop{\mathrm{E}}_{\beta}[\beta \in B \text{ and } u \leq \beta(u)], \qquad L_u = \mathop{\mathrm{E}}_{\beta}[\beta \in B \text{ and } u \leq \beta(w)],$$

$$\kappa_u = \prod_{\beta \in K_u} \beta, \qquad \lambda_u = \prod_{\beta \in L_u} \beta.$$

Furthermore, for each $u \in U$, let γ_u be the function on U to D such that

$$\gamma_u(u) = 1, \text{ and } \gamma_u(v) = 0 \text{ whenever } u \neq v \in U.$$

Finally, let

$$\xi = \sum_{u \in U} \kappa_u \text{ and } \eta = \sum_{u \in U} \gamma_u.$$

It is easy to see that

$$g^+(\kappa_u) = \lambda_u \text{ for any } u \in U$$

and hence

$$g^+(\xi) = \gamma_w.$$

Since $\gamma_w \in B$ and $\gamma_w(w) = 1$, it follows that

$$f^+(g^+(\xi)) = \mathrm{I}.$$

Letting

$$h = f[g],$$

we are going to show that $h^+(\xi) = \Theta$. In fact, for any closed element $\zeta \leq \xi$ we have $\zeta \leq \eta$ and, therefore,

$$\zeta \leq \sum_{u \in M} \gamma_u$$

for some finite subset M of U. Thus

$$\zeta \leq \xi \cdot \sum_{u \in M} \gamma_u = \sum_{u \in M} \kappa_u.$$

Since D was assumed to be infinite, there exists an element $a \in D$ such that $a \neq 1$ and $u \leq a$ for every $u \in M$. If δ is the function on U to D such that

$$\delta(u) = a \text{ for any } u \in U,$$

then

$$h^+(\zeta) \leq h(\delta) = \Theta.$$

Thus $h^+(\zeta) = \Theta$ whenever $\zeta \leq \xi$ and ζ is closed. Consequently $h^+(\xi) = \Theta$.

The results of this section can be generalized in another direction, namely by considering functions on one Boolean algebra into another Boolean algebra. Given two complete and atomistic Boolean algebras

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1\rangle \text{ and } \mathfrak{A}' = \langle A', +, 0, \cdot, 1\rangle,$$

and regular subalgebras

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1\rangle \text{ and } \mathfrak{B}' = \langle B' +, 0, \cdot, 1\rangle$$

of $\mathfrak{A}$ and $\mathfrak{A}'$ respectively, we can associate with any function f on B^m to B' a function f^+ on A to A' defined by the formula in Definition 2.1. The changes which have to be made in the formulation of Theorems 2.2-2.7 and 2.9 are quite obvious, and the proofs require no new ideas. The generalization of Theorem 2.8 involves six Boolean algebras, three complete and atomistic Boolean algebras $\mathfrak{A}$, $\mathfrak{A}'$, and $\mathfrak{A}''$ and regular subalgebras $\mathfrak{B}$, $\mathfrak{B}'$, and $\mathfrak{B}''$ of $\mathfrak{A}$, $\mathfrak{A}'$ and $\mathfrak{A}''$, respectively. The generalization of Theorem 2.10 is perhaps less obvious. Instead of the class Φ of functions described there we consider, for any Boolean algebras $\mathfrak{B} = \langle B, +, 0, \cdot, 1\rangle$ and

$\mathfrak{B}' = \langle B', +, 0, \cdot, 1\rangle$ and for any non-negative integer m, the class $\Phi_m(\mathfrak{B}, \mathfrak{B}')$ consisting of all functions f on B^m to B which are of the form $f = g[h_0, h_1, \cdots, h_{n-1}]$ where n is a non-negative integer, $h_0, h_1, \cdots, h_{n-1}$ are identity functions on B^m to B, and g is an additive function on B^m to B'. We then have:

Let $\mathfrak{A}$, $\mathfrak{A}'$, and $\mathfrak{A}''$ be perfect extensions of the Boolean algebras $\mathfrak{B}$, $\mathfrak{B}'$ and $\mathfrak{B}''$, respectively. If $g_0, g_1, \cdots, g_{n-1} \in \Phi_m(\mathfrak{B}, \mathfrak{B}')$ and $f \in \Phi_n(\mathfrak{B}', \mathfrak{B}'')$, then $f[g_0, g_1, \cdots, g_{n-1}] \in \Phi_m(\mathfrak{B}, \mathfrak{B}'')$ and

$$(f[g_0, g_1, \cdots, g_{m-1}])^+ = f^+[g^+_0, g^+_1, \cdots, g^+_{m-1}].$$

Observe that Theorems 1.23 and 1.24, as well as Theorems 2.16 and 2.17 that follow, are easy consequences of this result.

The generalization of Theorem 2.11 now offers no difficulty; the functions f, g, and h involved are assumed to belong to $\Phi_m(\mathfrak{B}, \mathfrak{B}')$. In order to generalize Theorem 2.12 we must first extend the notion of conjugate functions. A function g on B' to B will be called a conjugate of the function f on B to B' if, for every $x \in B$ and $y \in B'$, the formulas

$$f(x) \cdot y = 0 \text{ and } g(y) \cdot x = 0$$

are equivalent. Theorem 2.12 can then be extended to conjugate functions in this new sense.

It is interesting to notice that in the proofs of some of the theorems of this section—as well as in the proofs of their generalizations just indicated—the use of condition (iii) of Definition 1.19 can be avoided. This leads to the following results:

Consider two complete and atomistic Boolean algebras

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1\rangle \text{ and } \mathfrak{A}' = \langle A', +, 0, \cdot, 1\rangle$$

and let

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1\rangle \text{ and } \mathfrak{B}' = \langle B', +, 0, \cdot, 1\rangle$$

be subalgebras of $\mathfrak{A}$ and $\mathfrak{A}'$ respectively. We then have:

(i) If f is an additive function on B^m to B, then there exists an additive function g on A^m to A such that $g/B^m = f$.

(ii) If $\mathfrak{A}$ and $\mathfrak{B}$ satisfy condition (ii) of Definition 1.19, and if f is an additive function on B^m to B, then there exists a completely additive function g on A^m to B such that $g/B^m = f$.

To prove (ii) we define g as in the proof of Theorem 2.4. Then g is clearly a completely additive function with $g/B^m \leq f$. In order to show that $f \leq g/B^m$, we repeat with certain obvious changes the argument used in establishing formula (2) in the proof of Theorem 2.4.

Part (i) follows easily from (ii). In fact, there exists a complete and atomistic Boolean algebra $\mathfrak{A}_1 = \langle A_1, +, 0, \cdot, 1\rangle$ which is a perfect extension of $\mathfrak{A}$. Then condition (ii) of Definition 1.19 holds if we understand $\sum$ to denote summation over $\mathfrak{A}_1$, and it follows from (ii) that there exists a completely additive function h on $A_1{}^m$ to A' such that $h/B^m = f$. Consequently the function $g = h/A^m$ is additive and we have $g/B^m = f$.

One might ask what properties of the function f beside its additivity are preserved by the extension considered in (i) and in (ii). We have not studied this problem in detail, and shall only remark that if in (ii) we assume that $\mathfrak{A}$ is a perfect extension of $\mathfrak{B}$ and that f maps $\mathfrak{B}$ homomorphically into $\mathfrak{B}'$, then it can be shown that g maps $\mathfrak{A}$ homomorphically into $\mathfrak{A}'$. However, we omit the proof of this statement, since a much more general theorem of the same type is known from the literature.[7]

To conclude this section, we define formally the notion of a Boolean algebra with operators and we extend the notion of a perfect extension to this kind of algebras (Definitions 2.13 and 2.14). Using these new notions, we shall be able to summarize the main results so far achieved in a concise and convenient manner (Theorems 2.15 and 2.18).

DEFINITION 2.13. *By a Boolean algebra with operators we shall mean an algebra*

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, f_0, f_1, \cdots, f_\xi, \cdots\rangle$$

such that $\langle A, +, 0, \cdot, 1\rangle$ *is a Boolean algebra and the functions* f_ξ *are additive.*

Suppose $\mathfrak{A}$ *is a Boolean algebra with operators. By an atom of* $\mathfrak{A}$ *we mean an atom of the Boolean algebra* $\langle A, +, 0, \cdot, 1\rangle$. *We say that* $\mathfrak{A}$ *is atomistic if the Boolean algebra* $\langle A, +, 0, \cdot, 1\rangle$ *is atomistic. We call* $\mathfrak{A}$ *complete if the Boolean algebra* $\langle A, +, 0, \cdot, 1\rangle$ *is complete and if each of the operations* f_ξ *is completely additive. Finally, we say that* $\mathfrak{A}$ *is normal if each of the operations* f_ξ *is normal.*

DEFINITION 2.14. *Let*

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, f_0, f_1, \cdots, f_\xi, \cdots\rangle$$

[7] Cf. Sikorski [1], p. 332. Related problems are discussed in Sikorski [2].

and

$$\mathfrak{B} = \langle B, 0, +, \cdot, 1, g_0, g_1, \cdots, g_\xi, \cdots \rangle$$

be two Boolean algebras with operators. We say that $\mathfrak{A}$ *is a perfect extension of* $\mathfrak{B}$ *and that* $\mathfrak{B}$ *is a regular subalgebra of* $\mathfrak{A}$ *if the following conditions are satisfied:*

(i) $\mathfrak{A}$ *is complete and atomistic and* $\mathfrak{B}$ *is a subalgebra of* $\mathfrak{A}$.

(ii) $\langle B, +, 0, \cdot, 1 \rangle$ *is a regular subalgebra of* $\langle A, +, 0, \cdot, 1 \rangle$.

(iii) *For each of the functions* f_ξ *we have*

$$f_\xi(x) = \prod_{x \leqq y \in B^{m_\xi}} g_\xi(y) \quad \text{whenever } x \in At^{m_\xi},$$

where m_ξ *is the rank of* f_ξ *and* At *has its usual meaning.*

If $\mathfrak{A}$ is a perfect extension of $\mathfrak{B}$, we clearly have $f_\xi = g^+_\xi$.

THEOREM 2.15 (EXTENSION THEOREM). *For any Boolean algebra with operators* $\mathfrak{B}$ *there exists a complete and atomistic Boolean algebra with operators* $\mathfrak{A}$ *which is a perfect extension of* $\mathfrak{B}$.

Proof. Let

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1, g_0, g_1, \cdots, g_\xi, \cdots \rangle$$

be a Boolean algebra with operators. By 1.22 the Boolean algebra $\langle B, +, 0, \cdot, 1 \rangle$ is a regular subalgebra of a complete and atomistic Boolean algebra $\langle A, +, 0, \cdot, 1 \rangle$. If we put $f_\xi = g^+_\xi$ for each ξ, then, by 2.3, $\mathfrak{B}$ is a subalgebra of the algebra

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, f_0, f_1, \cdots, f_\xi, \cdots \rangle.$$

Moreover, by 2.4 and 2.13, $\mathfrak{A}$ is a complete and atomistic Boolean algebra with operators. Finally, 2.14(iii) is satisfied by 2.2. Hence $\mathfrak{A}$ is a perfect extension of $\mathfrak{B}$.

Just as in the case of ordinary Boolean algebras, the perfect extension $\mathfrak{A}$ is essentially determined by the Boolean algebra with operators $\mathfrak{B}$. This is shown in the next two theorems.

THEOREM 2.16. *Let*

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, f_0, f_1, \cdots, f_\xi, \cdots \rangle$$

and

$$\mathfrak{A}' = \langle A', +, 0, \cdot, 1, f'_0, f'_1, \cdots, f'_\xi, \cdots \rangle$$

be two complete and atomistic Boolean algebras with operators, and let

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1, g_0, g_1, \cdots, g_\xi, \cdots \rangle$$

and

$$\mathfrak{B}' = \langle A', +, 0, \cdot, 1, g'_0, g'_1, \cdots, g'_\xi, \cdots \rangle$$

be regular subalgebras of $\mathfrak{A}$ and $\mathfrak{A}'$, respectively. If a function ϕ maps $\mathfrak{B}$ isomorphically onto $\mathfrak{B}'$, then there exists a unique function ψ which maps $\mathfrak{A}$ isomorphically onto $\mathfrak{A}'$ in such a way that $\psi/B = \phi$.

Proof. By 1.23 there exists a unique function ψ such that

$$\psi/B = \phi \text{ and } \langle A, +, 0, \cdot, 1 \rangle \underset{\psi}{\cong} \langle A', +, 0, \cdot, 1 \rangle. \tag{1}$$

Let At be the set consisting of 0 and all the atoms of $\mathfrak{A}$. By (1) we have

$$\psi(x) = \sum_{x \geq u \in At} \prod_{u \leq y \in B} \phi(y) \text{ for any } x \in A. \tag{2}$$

Consider any operation f_ξ. If the rank of f_ξ is m, then, by 2.14,

$$f_\xi(x) = \sum_{x \geq u \in At^m} \prod_{u \leq y \in B^m} g_\xi(y) \text{ for any } x \in A^m.$$

Therefore, by (1) and (2),

$$\psi(f_\xi(x)) = \sum_{x \geq u \in At^m} \prod_{u \leq y \in B^m} g'_\xi(\phi(y_0), \phi(y_1), \cdots, \phi(y_{m-1}))$$

for any $x \in A^m$, and we easily conclude by (1) and 2.14 that

$$\psi(f_\xi(x)) = f'_\xi(\psi(x_0), \psi(x_1), \cdots, \psi(x_{m-1})) \text{ for any } x \in A^m.$$

Consequently,

$$\mathfrak{A} \underset{\psi}{\cong} \mathfrak{A}'.$$

THEOREM 2.17. *If*

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1, g_0, g_1, \cdots, g_\xi, \cdots \rangle$$

is a Boolean algebra with operators, and if $\mathfrak{A}$ and $\mathfrak{A}'$ are two perfect extensions of $\mathfrak{B}$, then there exists a unique function ψ mapping $\mathfrak{A}$ isomorphically onto $\mathfrak{A}'$ in such a way that $\psi(x) = x$ for every $x \in B$.

Proof. By 2.16.

THEOREM 2.18. *Let*

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, f_0, f_1, \cdots, f_\xi, \cdots \rangle$$

and

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1, g_0, g_1, \cdots, g_\xi, \cdots \rangle$$

be Boolean algebras with operators such that $\mathfrak{A}$ *is a perfect extension of* $\mathfrak{B}$. *If* L *is any equationally definable class of algebras such that* $\mathfrak{B}$ *belongs to* L, *then* $\mathfrak{A}$ *also belongs to* L.

Proof. By 2.2, 2.4, and 2.14 we have $f_\xi = g^+_\xi$ for each ξ. By 2.6, the same relation holds between the corresponding Boolean operations in $\mathfrak{A}$ and $\mathfrak{B}$; since $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, this also applies to 0 and 1 (if they are treated as operations with rank 0). If now h is an algebraic function over the similarity class to which $\mathfrak{A}$ and $\mathfrak{B}$ belong, then $h_\mathfrak{A}$ can be obtained by means of a definite process of iterated composition from the fundamental operations of $\mathfrak{A}$, and $h_\mathfrak{B}$ can be obtained by means of the same process from the corresponding operations of $\mathfrak{B}$. Hence, by 2.10.

$$h_\mathfrak{A} = h^+_\mathfrak{B}.$$

Similarly, if k is another algebraic function over the same similiarity class, we have

$$k_\mathfrak{A} = k^+_\mathfrak{B}.$$

Consequently, $h_\mathfrak{B} = k_\mathfrak{B}$ implies $h_\mathfrak{A} = k_\mathfrak{A}$; i. e., every equation between algebraic functions which holds (identically) in $\mathfrak{B}$ holds in $\mathfrak{A}$ as well. (For notions involved in this theorem and in its proof see Introduction.)

Section 3.

The Representation Theorem; Application to Closure Algebras and Cylindric Algebras.

A set-field B whose universal set is U will be called *regular* if the Boolean algebra $\langle \mathsf{B}, \cup, -, \cap, U \rangle$ is a regular subalgebra of the Boolean algebra $\langle \mathsf{A}, \cup, -, \cap, U \rangle$, where A is the family of all subsets of U. The representation theorem for Boolean algebras can then be stated as follows:

THEOREM 3.1. *Given any Boolean algebra*

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1 \rangle,$$

there exists a regular set-field B' *such that* $\mathfrak{B}$ *is isomorphic to the Boolean algebra*

$$\mathfrak{B}' = \langle \mathsf{B}', \cup, \Lambda, \cap, U \rangle,$$

where U *is the universal set of* B'.

As a consequence of this theorem, the study of Boolean algebras reduces to the study of regular set-fields. We shall therefore proceed to investigate additive functions over regular set-fields. This will lead us to a representation theorem for Boolean algebras with operators (Theorem 3.10). In the second part of this section we apply the extension theorem and the representation theorem to closure algebras and cylindric algebras.

Throughout the first part of this section we fix a regular set-field B, let U be the universal set of B, and let A be the family of all subsets of U. In accordance with our previous convention, At will denote the set consisting of the zero element and all the atoms of the Boolean algebra $\langle \mathsf{A}, \cap, \Lambda, \cup, U \rangle$, i. e., At consists of the empty set Λ and all sets of the form $\{x\}$ with $x \in U$.

We begin by establishing a one-to-one correspondence between all normal and completely additive functions on A^m to A and all $(m+1)$-ary relations $R \subseteq U^{m+1}$.

DEFINITION 3.2. *If* $R \subseteq U^{m+1}$, *then the function* R^* *on* A^m *to* A *is defined by the formula*

$$R^*(X) = \mathop{\mathrm{E}}_{y} [x^\frown y \in R \text{ for some } x \in X_0 \times X_1 \times \cdots \times X_{m-1}] \text{ for any } X \in \mathsf{A}^m.$$

In this definition the last terms of the sequences belonging to R play a special role. This may be somewhat unnatural in case R is not a function and we would of course correlate with R m other functions on A^m to A besides R^*. However, these functions will not be needed in the present work.

THEOREM 3.3. *If* $R \subseteq U^{m+1}$, *then* R^* *is a normal and completely additive function on* A^m *to* A.

Conversely, if F *is a normal and completely additive function on* A^m *to* A, *then there exists a unique relation* $R \subseteq U^{m+1}$ *such that* $F = R^*$. *This relation is defined by the formula*

$$R = \mathop{\mathrm{E}}_{x^\frown y} [x \in U^m \text{ and } y \in F(\{x_0\}, \{x_1\}, \cdots, \{x_{m-1}\})].^{8}$$

[8] Theorem 3.3 (restricted to functions of rank 1) is stated without proof in Tarski [3]. The same applies to Theorem 3.6 and to that part of Theorem 3.5 which concerns the equivalence of (i_4) and (ii_4).

Proof. The first part follows from 1.6(i′)(ii′) and 3.2. To prove the second part, suppose F is a normal and completely additive function on A^m to A, and let R be defined as above. Since R^* is a normal and completely additive function on A^m to A by the first part of the theorem, we see by 1.8(ii) that $F = R^*$ is equivalent to the formula

(1) $$F(X) = R^*(X) \text{ whenever } X \in \mathsf{At}^m.$$

If $X_j = \Lambda$ for some $j < \mathrm{m}$, then this follows from the normality of F and R^*. If $X_j \neq \Lambda$ for every $j < m$, then there exists a sequence $x \in U^m$ such that

$$X_j = \{x_j\} \text{ for every } j < m.$$

For any element $y \in U$ we then have

$$y \in F(X) \text{ if and only if } x^\frown y \in R.$$

By 3.2 we have

$$x^\frown y \in R \text{ if and only if } y \in R^*(X),$$

and we conclude that (1) holds in this case also. The uniqueness of R is an immediate consequence of 3.2.

THEOREM 3.4. *If F is a normal and additive function on B^m to B, then there exists a relation $R \subseteq U^{m+1}$ such that $F = R^*/\mathsf{B}^m$.*

Proof. By 2.3, 2.4, and the hypothesis, F^+ is a normal and completely additive function with $F = F^+/\mathsf{B}^m$. Hence, by 3.3, there exists a relation $R \subseteq U^{m+1}$ such that $F^+ = R^*$ and, consequently, $F = R^*/\mathsf{B}^m$.

It follows from 3.3 and 3.4 that given a normal and completely additive function F on A^m to A, or a normal and additive function F on B^m to B, we can express any property that F may possess as a property of the corresponding relation R. In the next three theorems we provide some examples of how this can be done. In order to state these theorems, we shall need certain familiar notions from the calculus of binary relations. The *domain* of a binary relation R is the set consisting of all elements x such that $\langle x, y\rangle \in R$ for some element y; the *field* of R is the union of the domain of R and the domain of R^{-1}. A binary relation R is said to be *reflexive* if $\langle x, x\rangle \in R$ whenever x is in the field of R, *symmetric* if $R^{-1} \subseteq R$, and *transitive* if $R|R \subseteq R$. If R is both symmetric and transitive (and hence also reflexive), then we call it an *equivalence relation.*

THEOREM 3.5. *Let F be a function on A to A. Assume that F is normal and completely additive and satisfies one or more of the following conditions:*

(i_1) $X \subseteq F(X)$ *for any* $X \in \mathsf{A}$,

(i_2) $F(F(X)) \subseteq F(X)$ *for any* $x \in \mathsf{A}$,

(i_3) F *is selfconjugate,*

(i_4) $F(X \cap Y) = F(X) \cap F(Y)$ *for any* $X, Y \in \mathsf{A}$,

(i_5) $F(F(X)) = X \cap F(U)$ *for any* $X \in \mathsf{A}$,

(i_6) $F(F(X)) = X$ *for any* $X \in \mathsf{A}$.

Then there exists a relation $R \subseteq U^2$ *with* $F = R^*$ *which satisfies the corresponding condition, or conditions, in the following list:*

(ii_1) R is *reflexive and the domain of* R *is* U,

(ii_2) R *is transitive,*

(ii_3) R *is symmetric,*

(ii_4) R^{-1} *is a function,*

(ii_5) R *is symmetric and is a function,*

(ii_6) R *is symmetric and is a function whose domain is* U.

Conversely, if there is a relation $R \subseteq U^2$ *with* $F = R^*$ *which satisfies one or more of the conditions* (ii_1)-(ii_6), *then* F *is normal and completely additive and satisfies the corresponding conditions* (i_1)-(i_6).

Proof. By 3.3, F is normal and completely additive if and only if $F = R^*$ for some $R \subseteq U^2$. Assuming that $F = R^*$ we shall prove that the conditions (i_5) and (ii_5) are equivalent (assuming that the equivalence of (i_3) and (ii_3) has already been established); the corresponding proofs for the remaining conditions will be omitted.

Assume that (i_5) holds. For any element $x \in U$ we then have

(1) $F(F(\{x\})) = \{x\}$ if $x \in F(U)$, and $F(F(\{x\})) = \Lambda$ if $x \notin F(U)$.

Consider two elements $x, y \in U$ with $\langle x, y \rangle \in R$. Then, by 3.2, $y \in F(\{x\})$ whence, by the additivity of F, $y \in F(U)$. By (1) we therefore have $F(F(\{y\})) = \{y\}$ and, F being normal, this implies that $F(y) \neq \Lambda$. Consequently, by the additivity of F, $F(F(\{x\})) \neq \Lambda$. By (1) this implies that $F(F\{x\})) = \{x\}$ and hence $F(\{y\}) = \{x\}$. We therefore see by 3.2 that $\langle y, x \rangle \in R$. Thus R is symmetric. To show that R is a function, consider two elements $x, y \in U$ such that $\langle y, x \rangle \in R$. Since R is symmetric, we have $\langle x, y \rangle \in R$; as we have already shown, this leads to the conclusion $F(\{y\}) = \{x\}$. Together with 3.2 this shows that for each element $y \in U$ there is at most one element $x \in U$ such that $\langle y, x \rangle \in R$. Hence R is a function and (ii_5) holds.

If (i_5) is satisfied, and if $x, y \in U$ are such that $y \in F(F(\{x\}))$, then, by 3.2, there exists an element $z \in U$ such that $\langle x, z\rangle \in R$ and $\langle z, y\rangle \in R$. Since R is symmetric and is a function, this implies that $\langle z, x\rangle \in R$ and, hence, $x = y$. Thus $F(F(\{x\})) \subseteq \{x\}$ for any $x \in U$, and, consequently, by 1.8(1), $F(F(X)) \subseteq X$ for any $X \in A$. Therefore, by the additivity of F,

$$X \cap F(U) = F(F(X)) \cup [X \cap F(F(X)^-)].$$

Since (i_3) is satisfied, it follows that (i_5) holds.

THEOREM 3.6. *Two functions F and G on A to A are conjugate if, and only if, there exists a relation $R \subseteq U^2$ such that*

$$F = R^* \text{ and } G = (R^{-1})^*.$$

Proof. If F and G are conjugate, then F and G are normal and completely additive by 1.14. It follows by 3.3 that there exist relations $R, S \subseteq U^2$ such that $F = R^*$ and $G = S^*$. Consider two elements $x, y \in U$ with $\langle x, y\rangle \in R$. Then, by 3.2, $y \in F(\{x\})$ or, equivalently, $F(\{x\}) \cap \{y\} \neq \Lambda$. By 1.11 this gives $G(\{y\}) \cap \{x\} \neq \Lambda$, so that $x \in G(\{y\})$ and, consequently, $\langle y, x\rangle \in S$. We thus have $R^{-1} \subseteq S$. Similarly $S^{-1} \subseteq R$, and we conclude that $S = R^{-1}$.

Assume that $F = R^*$ and $G = (R^{-1})^*$. Consider two sets $X, Y \subseteq U$ with $F(X) \cap Y = \Lambda$. For any elements $x \in X$ and $y \in Y$ we then have $\langle x, y\rangle \notin R$ and, hence, $\langle y, x\rangle \notin R^{-1}$. It follows that $G(Y) \cap X = \Lambda$. Similarly, if $X, Y \subseteq U$ are such that $G(Y) \cap X = \Lambda$, then $F(X) \cap Y = \Lambda$. Hence F and G are conjugate by 1.11.

THEOREM 3.7. *Theorem 3.5 remains valid if we replace there 'A' by 'B,' 'completely additive' by 'additive' and 'R^*' by 'R^*/B.' Similarly Theorem 3.6 remains valid if we replace there 'A' by 'B,' 'R^*' by 'R^*/B,' and '$(R^{-1})^*$' by '$(R^{-1})^*/\mathsf{B}$.'*

Proof. Suppose F is a normal and additive function on B to B. If F satisfies one or more of the conditions 3.5 (i_1)-(i_6) with 'A' replaced by 'B,' then, by 2.18, the same condition will be satisfied if we replace 'F' by 'F^+' but leave 'A' unchanged. Since, by 2.3 and 2.4, F^+ is normal and completely additive, there exists, by 3.5, a relation $R \subseteq U^2$ which satisfies the corresponding conditions 3.5(i_1)-(ii_6) and for which $R^* = F^+$ and, hence, $R^*/\mathsf{B} = F$.

If, conversely, there exists a relation $R \subseteq U^2$ such that $F = R^*/\mathsf{B}$, then F is normal and additive by 3.3. If, in addition, R satisfies one or more of

the conditions 3.5(ii$_1$)-(ii$_6$), then, by 3.3 and 3.5, the corresponding conditions 3.5(i$_1$)-(i$_6$) will be valid if we replace there 'F' by 'R^*.' Consequently the same conditions will hold if we leave F unchanged but replace 'A' by 'B.' The proof of the second part of the theorem is similar.

DEFINITION 3.8. *By the complex algebra of an algebra in the wider sense*

$$\mathfrak{U} = \langle U, R_0, R_1, \cdots, R_\xi, \cdots \rangle$$

we mean the algebra

$$\mathfrak{A} = \langle \mathsf{A}, \cup, \wedge, \cap, U, R^*_0, R^*_1, \cdots, R^*_\xi, \cdots \rangle$$

where A *is the family of all subsets of* U.

THEOREM 3.9. *The complex algebra of any algebra in the wider sense is a normal, complete, and atomistic Boolean algebra with operators.*

Conversely, every normal, complete, and atomistic Boolean algebra with operators is isomorphic to the complex algebra of some algebra in the wider sense.

Proof. The first part follows from 2.13, 3.3, and 3.8. Suppose

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, f_0, f_1, \cdots, f_\xi, \cdots \rangle$$

is a normal, complete, and atomistic Boolean algebra with operators. Let the rank of f_ξ be m_ξ. Then $\mathfrak{A}$ is isomorphic to an algebra

$$\mathfrak{A}' = \langle \mathsf{A}, \cup, \wedge, \cap, U, F_0, F_1, \cdots, F_\xi, \cdots \rangle$$

where U is some set, A is the family of all subsets of U, and F_ξ is a normal and completely additive function on A^{m_ξ} to A. By 3.33 there exist relations $R_0, R_1, \cdots, R_\xi, \cdots$ such that $R_\xi \subseteq U^{m_\xi+1}$ and $F_\xi = R^*_\xi$. Hence, by 3.8, $\mathfrak{A}'$ is the complex algebra of the algebra in the wider sense

$$\mathfrak{U} = \langle U, R_0, R_1, \cdots, R_\xi, \cdots \rangle.$$

THEOREM 3.10 (REPRESENTATION THEOREM). *Every normal Boolean algebra with operators is isomorphic to a regular subalgebra of the complex algebra of some algebra in the wider sense.*

Proof. If $\mathfrak{B}$ is a normal Boolean algebra with operators, then there exists, by 2.15, a complete and atomistic Boolean algebra with operators $\mathfrak{A}$

which is a perfect extension of $\mathfrak{B}$. By 2.13 and 2.14, $\mathfrak{A}$ is also normal, and the conclusion follows by 3.9.

We now turn to the applications of our results to closure algebras and cylindric algebras.

DEFINITION 3.11. *An algebra*

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, C \rangle$$

(where $+$ and $\cdot$ are operations on A^2 to A, C is an operation on A to A, and 0 and 1 are elements of A) is called a closure algebra if it satisfies the following conditions:

(i) $\langle A, +, 0, \cdot, 1 \rangle$ *is a Boolean algebra.*

(ii) C *is an additive function on A to A.*

(iii) $C(C(x)) = C(x)$ *for any $x \in A$.*

(iv) $x + C(x) = C(x)$ *for any $x \in A$.*

(v) $C(0) = 0$.

Since each of the conditions (iii)-(v) has the form of an equation, we can apply Theorem 2.15 to closure algebras. This gives

THEOREM 3.12. *For any closure algebra $\mathfrak{B}$ there exists a complete and atomistic closure algebra $\mathfrak{A}$ which is a perfect extension of $\mathfrak{B}$.*

Proof. By 2.15, 2.18, and 3.11.

Before applying Theorem 3.10 to closure algebras, we ask when the complex algebra of a given algebra in the wider sense is a closure algebra. The answer to this question is given in the following

THEOREM 3.13. *Suppose U is an arbitrary set, A is the family of all subsets of U, and $R \subseteq U^2$. Then the algebra*

$$\mathfrak{A} = \langle \mathsf{A}, \cup, -, \cap, U, R^* \rangle$$

is a closure algebra if, and only if, R is reflexive and transitive, and the domain of R is U.[9]

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[9] This theorem is essentially known from the literature; compare Birkhoff [1], Theorem 1.

Proof. Since, by 3.3, R^* is normal and completely additive, we see by 3.11 that $\mathfrak{A}$ is a closure algebra if, and only if,

$$X \subseteq R^*(X) \text{ and } R^*(R^*(X)) \subseteq R^*(X) \text{ for any } X \in \mathsf{A}.$$

Hence the conclusion follows from 3.3 and 3.5.

THEOREM 3.14. *An algebra*

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1, C \rangle$$

is a closure algebra if, and only if, $\mathfrak{B}$ is isomorphic to an algebraic system

$$\mathfrak{B}' = \langle \mathsf{B}, \cup, \wedge, \cap, U, R^* \rangle$$

where B is a regular set-field whose universal set is U, and R is a reflexive and transitive relation whose domain is U and which satisfies the condition: $R^*(X) \in \mathsf{B}$ for every $X \in \mathsf{B}$ (i. e., R^*/B is an operation on B to B).

Proof. Assume that $\mathfrak{B}$ is a closure algebra. By 3.12, there is a complete and atomistic closure algebra $\mathfrak{A}$ which is a perfect extension of $\mathfrak{B}$. By 3.8, 3.9, 3.11, and 3.13, $\mathfrak{A}$ is isomorphic to an algebra

$$\mathfrak{A}' = \langle \mathsf{A}, \cup, \wedge, \cap, U, R^* \rangle$$

where U is a certain set, A is the family of all subsets of U, and R is a reflexive and transitive relation whose domain is U. By 2.14, $\mathfrak{B}$ is a regular subalgebra of $\mathfrak{A}$, and hence it is isomorphic to a regular subalgebra

$$\mathfrak{B}' = \langle \mathsf{B}, \cup, \wedge, \cap, U, R^* \rangle$$

of $\mathfrak{A}'$. Since, by 3.11, $\mathfrak{B}$ is a Boolean algebra supplemented by a unary operation, B is a regular set-field, and R^*/B is an operation on B to B.

If, conversely, $\mathfrak{B}$ is isomorphic to an algebra $\mathfrak{B}'$ with the properties listed in the theorem, then, by 3.13, $\mathfrak{B}$ is isomorphic to a subalgebra of a closure algebra and hence, by 3.11, it is itself a closure algebra.

The theorem just proved clearly remains valid if we omit in it the term 'regular.' An analogous remark applies to Theorem 3.18 below.

By a representation theorem for closure algebras which is known from the literature, every closure algebra $\mathfrak{B}$ is isomorphic to an algebra

$$\mathfrak{B}' = \langle \mathsf{B}, \cup, \wedge, \cap, U, C \rangle$$

where U is a topological space with the closure operation C, and B is a set-field whose universal set is U and which satisfies the condition: $C(X) \in \mathfrak{B}$ for every $X \in \mathsf{B}$; in addition, in this topological space we have $C(X) = X$ for every set $X \subseteq U$ consisting of just one point.[10] On the other hand, Theorem 3. 14 implies that every closure algebra $\mathfrak{B}$ is isomorphic to an algebra $\mathfrak{B}'$ as described above, in which, however, we do not necessarily have $C(X) = X$ for every one-point set $X \subseteq U$, but in which instead the closure operation C is complete additive. The new representation theorem appears to have interesting implications for every topological space whose closure operation is not completely additive; for it shows that the closure algebra of all subsets of such a space can be isomorphically embedded in a topological space with a completely additive closure operation.

DEFINITION 3. 15. *An algebra*

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, C_0, C_1 \rangle$$

(where $+$ and $\cdot$ are operations on A^2 to A, C_0 and C_1 are operations on A to A, and 0 and 1 are elements of A) is called a cylindric algebra if the following conditions are satisfied:

(i) $\langle A, +, 0, \cdot, 1 \rangle$ *is a Boolean algebra.*

(ii) C_0 *and* C_1 *are selfconjugate functions (i. e., for any elements* $x, y \in A$, *if* $C_0(x) \cdot y = 0$, *then* $C_0(y) \cdot x = 0$, *and if* $C_1(x) \cdot y = 0$, *then* $C_1(y) \cdot x = 0$).

(iii) $C_0(C_0(x)) = C_0(x)$ *and* $C_1(C_1(x)) = C_1(x)$ *for any* $x \in A$.

(iv) $C_0(C_1(x)) = 1$ *for any element* $x \in A$ *with* $x \neq 0$.[11]

THEOREM 3. 16. *For any cylindric algebra $\mathfrak{B}$ there exists a complete and atomistic cylindric algebra $\mathfrak{A}$ which is a perfect extension of $\mathfrak{B}$.*

Proof. Suppose that

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1, C_0, C_1 \rangle$$

is a cylindric algebra. By 1. 14, 2. 13, and 3. 15, $\mathfrak{B}$ is a Boolean algebra with

[10] Cf. McKinsey-Tarski [1], Theorem 2. 6.

[11] A slightly different, though equivalent, definition of these algebras is given in Chin-Tarski [2] (where, however, the term "cylindric algebra" is not used).

operators. Hence, by 2.15, there exists a complete and atomistic Boolean algebra with operators

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, f_0, f_1 \rangle$$

which is a perfect extension of $\mathfrak{B}$. Thus $\mathfrak{A}$ satisfies 3.15(i). By 2.2, 2.4, 2.13, and 2.14 we have $f_0 = C^+_0$ and $f_1 = C^+_1$; hence, by 2.10 and 2.12, $\mathfrak{A}$ satisfies 3.15(ii)(iii). Finally, by arguing as in the proof of 2.11(ii), we show that $\mathfrak{A}$ satisfies 3.15(iv). Consequently, $\mathfrak{A}$ is a cylindric algebra.

THEOREM 3.17. *Suppose U is an arbitrary set, A is the family of all subsets of U, $R_0 \subseteq U^2$, and $R_1 \subseteq U^2$. Then the algebra*

$$\mathfrak{A} = \langle \mathsf{A}, \cup, \wedge, \cap, U, R^*_0, R^*_1 \rangle$$

is a cylindric algebra if, and only if, the following conditions are satisfied:

(i) *R_0 and R_1 are equivalence relations.*

(ii) $R_0 | R_1 = U^2$.

Proof. If $\mathfrak{A}$ is a cylindric algebra, then R_0 and R_1 are equivalence relations by 3.3, 3.5, and 3.15(ii)(iii). By 3.2 and 3.15(iv) we have $R_1 | R_0 = U^2$ and, hence, $R_0 | R_1 = (R_1 | R_0)^{-1} = U^2$. Thus (i) and (ii) hold.

If (i) and (ii) are satisfied, then we see by 3.2 that 3.15(iv) holds (with the obvious substitutions) while conditions (ii) and (iii) of 3.15 follow from 3.3 and 3.5 and the fact that R_0 and R_1 are transitive.

THEOREM 3.18. *An algebra*

$$\mathfrak{B} = \langle B, +, 0, \cdot, 1, C_0, C_1 \rangle$$

is a cylindric algebra if, and only if, $\mathfrak{B}$ is isomorphic to an algebraic system

$$\mathfrak{B}' = \langle \mathsf{B}, \cup, \wedge, \cap, U, R^*_0, R^*_1 \rangle$$

satisfying the following conditions:

(i) *B is a regular set-field; U is the universal set of B; $R^*_0(X) \in \mathsf{B}$ and $R^*_1(X) \in \mathsf{B}$ whenever $X \in \mathsf{B}$.*

(ii) *R_0 and R_1 are equivalence relations.*

(iii) $R_0 | R_1 = U^2$.

Proof. Analogous to that of 3.14, with 3.11, 3.12, and 3.13 replaced by 3.15, 3.16, and 3.17, respectively.

An important modification of Theorem 3.18 is known (which, however, requires an essentially new method of arguing).[12] It can be shown that, after omitting the term "regular" in condition (i), the following condition can be added:

(iv) $R_0 \cap R_1$ is the identity function on U to U.

The result thus obtained admits an interesting geometric interpretation. For, as is easily seen, it implies that the set U and the relations R_0 and R_1 can be chosen in 3.18 in such a way that: (i) U is a set of the form $U = X \times Y$, and hence can be regarded as an abstract two-dimensional analytic space with the axes X and Y; (ii) R_0 is the relation which holds between any two couples $\langle x, y\rangle$ and $\langle x', y'\rangle$ in U if, and only if, $y = y'$, and hence R^*_0 coincides with the so-called operation of cylindrification parallel to the axis X; similarly, (iii) R_1 holds between $\langle x, y\rangle, \langle x', y'\rangle \in U$ if, and only if, $x = x'$, and hence R^*_1 coincides with cylindrification parallel to Y. In this geometric form, the modified Theorem 3.18 motivates the choice of the name given cylindric algebras, and it becomes the natural representation theorem for this class of algebraic systems.

It can readily be shown that every cylindric algebra $\mathfrak{A}$ satisfies the condition

(iv') $C_0(C_1(x)) = C_1(C_0(x))$ for every $x \in A$.

If we replace (iv) by (iv') in 3.15, we obtain the definition of a more comprehensive class of algebras, which can be referred to as *generalized cylindric algebras*. Cylindric algebras as originally defined prove to coincide with those generalized cylindric algebras which are simple (in the general algebraic sense). Theorems 3.16-3.18 extend to generalized cylindric algebras; obviously, formulas 3.17(ii) and 3.18(iii) have to be replaced by $R_0 \mid R_1 = R_1 \mid R_0$. Theorem 3.18 can again be improved so as to admit a geometric interpretation. In this interpretation the set U is assumed to coincide, not necessarily with the whole two-dimensional space $X \times Y$, but with a subset of $X \times Y$ (subjected to some additional conditions), while R_0 and R_1 preserve their old meaning.

On the other hand, *projective algebras* (mentioned in the introduction) can be regarded as specialized cylindric algebras. In fact, a projective algebra can be characterized as an algebraic system

$$\mathfrak{A} = \langle A, +, 0, \cdot, 1, C_0, C_1, \Theta\rangle$$

[12] In fact, the method used in Everett-Ulam [1], Section 4, pp. 85-88, can be applied here.

satisfying the conditions:

(i) $\langle A, +, 0, \cdot, 1, C_0, C_1\rangle$ is a cylindric algebra,

(ii) Θ is an atom in A,

(iii) $x = C_0(x) \cdot C_1(x)$ whenever $x \in A$ and $x \leq C_0(\Theta)$ or $x \leq C_1(\Theta)$.

In terms of C_0, C_1, and Θ two further operations on A to A can be defined:

$$P_0(x) = C_0(x) \cdot C_1(\Theta) \text{ and } P_1(x) = C_1(x) \cdot C_0(\Theta) \text{ for every } x \in A.$$

Conversely, C_0, C_1, and Θ can be re-defined in terms of P_0 and P_1; hence, projective algebras can also be characterized as Boolean algebras supplemented by these two new operations. Again Theorems 3.16-3.18 extend, with obvious changes, to projective algebras; and the extended Theorem 3.18 can be improved so as to admit a geometric interpretation (and thus to yield the natural representation theorem for these algebras). In this interpretation U, R_0, and R_1 have the same meaning as in the case of cylindric algebras. Θ is represented as the set consisting of a single couple $\langle x_0, y_0\rangle$ for some $x_0 \in X$ and $y_0 \in Y$; this couple can be regarded as the origin of the coordinate system in the space $U = X \times Y$. Finally, P_0 and P_1 are represented as the operations of projection parallel to the axes X and Y.[13]

BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND,
UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA.

[13] Our definition of projective algebras is essentially equivalent to the one given in Everett-Ulam [1] where, however, P_0 and P_1 (instead of C_0, C_1 and Θ) are treated as fundamental operations. A definition of projective algebras in terms of C_0, C_1, and Θ can be found in Chin-Tarski [2]; however, Mr. F. B. Thompson has pointed out to us that in this definition the postulates involving Θ are too weak and should be modified in the way indicated in the text. The improved Theorem 3.18 in its geometric form, when extended to projective algebras, essentially coincides with the representation theorem which has been established in Everett-Ulam [1] for complete atomistic projective algebras and generalized in McKinsey [1] to all projective algebras.

ORTHOGONAL AND UNITARY GROUPS OVER THE RATIONAL FIELD.*

By JEAN DIEUDONNÉ.

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1. Let E be an n-dimensional space over the field $\boldsymbol{Q}$ of rational numbers; let f be a non-degenerate symmetric bilinear form defined in E, and $g(x) = f(x, x)$ the corresponding quadratic form. In a recent paper [2],[1] I have proved that for $n \geqq 7$, the commutator subgroup $P\Omega_n(\boldsymbol{Q}, f)$ of the projective orthogonal group $\boldsymbol{PO}_n(\boldsymbol{Q}, f)$ is always *simple*, even when the index ν of f is 0. By a modification of the argument used in the proof of that theorem, I shall now show that the result is still true for $n = 6$. Using the well-known isomorphism between orthogonal groups on 6 variables and unitary groups on 4 variables, this enables me to prove that if K_1 is any quadratic extension of $\boldsymbol{Q}$, E_1 an n-dimensional space over K_1, f_1 a non-degenerate hermitian form over E_1, the commutator subgroup of the projective unitary group $\boldsymbol{PU}^+{}_n(K_1, f_1)$ is *simple* for $n \geqq 5$, even when the index of f_1 is 0.

2. The proof given in [2] uses essentially a theorem of Meyer to the effect that a non-degenerate quadratic form in at least 4 variables over the rational field can take *any* rational value if it is indefinite, any positive (resp. negative) rational value if it is positive definite (resp. negative definite). In applying that theorem, I have to use repeatedly the assumption $n \geqq 6$; however, the assumption $n \geqq 7$ is used essentially only twice, in steps 3° and 4° of the proof. I proceed to show that, by a modification of the argument, these steps too can be carried out for $n = 6$. I keep the notation of [2]. G being a normal subgroup of $\Omega_n(\boldsymbol{Q}, f)$ not contained in the center, step 1° of the proof shows that there exists in G a plane rotation u distinct from the identity. If u^2 is the identity, u is an $(n-2, 2)$-involution in Ω_n, and therefore [2, n° 10] a special involution; in that case, step 3° is proved. If u is not an involution, let P be its plane, a and b two orthogonal vectors in P, and let $u(a) = \lambda a + \mu b$; let $\alpha = g(a)$, $\beta = g(b)$ (which are positive rational numbers). If E' is the space over the real field $\boldsymbol{R}$ obtained from E by extension of the field of scalars, u is extended to a plane rotation in E', of an

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* Received May 14, 1951.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

angle θ such that $\cos\theta = \lambda$, $\sin\theta = \mu(\beta/\alpha)^{\frac{1}{2}}$; replacing if necessary u by a power u^m of u, it can always be supposed that $\pi/2 < \theta < \pi$ [2, n° 3]. Consider the vector $c = \frac{1}{2}(a + w(a)) = \frac{1}{2}((1+\lambda)a + \mu b)$; it is easily verified that $u = s's$, where s is the symmetry with respect to the hyperplane orthogonal to a, and s' the symmetry with respect to the hyperplane orthogonal to c. We are going to show that there exists a vector x such that $g(x) = g(c)$, $f(a,x) = f(a,c)$ and such that x and c are orthogonal, that is, $f(x,c) = 0$. To that purpose, let y be a vector in the hyperplane orthogonal to a, and write $x = \frac{1}{2}((1+\lambda)a + \mu y)$; this already insures that $f(a,c) = f(a,x)$. The relation $f(x,c) = 0$ then becomes $\alpha(1+\lambda)^2 + \mu^2 f(b,y) = 0$; write $y = \rho b + z$, where $z \in P^*$ (orthogonal to both a and b); this yields

$$\rho = -(\alpha/\beta)(1+\lambda)^2/\mu^2 = -(1+\cos\theta)^2/\sin^2\theta = -\operatorname{cotg}^4\theta/2$$

Finally, the relation $g(x) = g(c)$ is equivalent to $g(y) = g(b)$, which in turn gives $g(z) = (1-\rho^2)g(b)$. Now, as $\pi/4 < \theta/2 < \pi/2$, one has $|\operatorname{cotg}\theta/2| < 1$, and therefore $\rho^2 < 1$; as P^* is at least 4-dimensional, Meyer's theorem shows that a vector z satisfying the preceding relation can be found in P^*.

It follows from Witt's theorem that there exists an orthogonal transformation $v \in \boldsymbol{O}_n$ such that $v(a) = a$ and $v(c) = x$; the argument in [2, n° 5] (which only uses the assumption $n \geqq 6$) proves then that there exists a transformation $w \in \boldsymbol{O}_n$ leaving invariant a and x, and such that $v_1 = wv$ is in the commutator subgroup Ω_n. The transformation $u_1 = wv_1u^{-1}v_1^{-1}$ belongs therefore to G; but $v_1u^{-1}v_1^{-1} = ss''$, where $s'' = v_1s'v_1^{-1}$ is the symmetry with respect to the hyperplane orthogonal to x. Therefore $u_1 = s'sss'' = s's''$ is a special involution in G (since x and c are orthogonal), and this proves point 3°.

3. We now prove point 4°, which states that G contains *every* special involution. To achieve this, we notice that, in the preceding construction, we had $g(c) = \frac{1}{4}((1+\lambda)^2\alpha + \mu^2\beta) = \frac{1}{2}\alpha(1+\lambda) = \frac{1}{2}f(a, a+u(a))$. All we have to do is therefore to find a plane rotation u in G and a vector z in its plane such that $f(z,u(z)) < 0$ and that $\frac{1}{2}f(z, z+u(z))$ is equal to a given positive rational number γ_0 multiplied by a square. We start from a special involution w in G, of plane R; let b be a vector in R, and consider all vectors $z \in E$ of the form $z = a + b$, where $a \in R^*$; we write $\alpha = f(a,a)$, $\beta = f(b,b)$. For any such vector z, there exists in R^* (which is at least 4-dimensional) a vector y such that $f(y,y) = f(z,z)$, by Meyer's theorem. Let s, s' be the symmetries with respect to the hyperplane orthogonal to z and y; they are conjugate in $\boldsymbol{O}_n$ by Witt's theorem, hence $v = s's$ is in Ω_n.

It follows that $u = wv^{-1}w^{-1}v$ is in G; but as $w(y) = y$, $wv^{-1}w^{-1} = s''s'$, where s'' is the symmetry with respect to the hyperplane orthogonal to $w(z) = a - b$; therefore $u = s''s$ is a plane rotation in G, z is in its plane, and $u(z) = -s''(z) = -z + 2f(z, w(z))w(z)/g(z)$. This formula shows that the conditions we have to satisfy are $(\alpha - \beta)^2/(\alpha + \beta) = \gamma_0\xi^2$, where $\xi \in \boldsymbol{Q}$, and $\alpha + \beta > 2^{\frac{1}{2}}(\alpha - \beta)$. Now it is easily verified that the first requirement is satisfied by taking $\xi = t - 2\beta/(\gamma_0 t)$, $\alpha + \beta = \gamma_0 t^2$, $\alpha - \beta = \gamma_0\xi t$, where t is any positive rational number; the condition $\alpha > 0$ reduces to $t^2 > \beta/\gamma_0$, and the condition $\alpha + \beta > 2^{\frac{1}{2}}(\alpha - \beta)$ to $t^2 < 2(2^{\frac{1}{2}})/(2^{\frac{1}{2}} - 1) \cdot \beta/\gamma_0$. These conditions are therefore compatible, and with a value of t which satisfies them, the rotation u will satisfy our requirements, provided there is a vector $a \in R^*$ such that $f(a, a) = \alpha$; but since $\alpha > 0$ and R^* is at least 4-dimensional, the existence of a is proved by Meyer's theorem.

We have thus proved the existence in G of a special involution conjugate (within $\boldsymbol{O}_n$) of any special involution. The end of the proof [2, end of n° 5 and n° 6] only makes use of the relation $n \geqq 6$, and we see therefore that *for any form f of index 0 in E, the group $P\Omega_6(\boldsymbol{Q}, f)$ is simple.*

Of course, the remarks made in [2, n° 7] still apply here, and $P\Omega_6(K, f)$ is simple for any subfield K of $\boldsymbol{R}$ in which Meyer's theorem is valid. For the field $\boldsymbol{Q}$, the theorem is known to be false for $n = 3$ and $n = 4$ [1, p. 38]; the only case which remains open in that case is $n = 5$; an entirely new approach seems to be needed for that case, since the assumption $n \geqq 6$ is essential to all but the first two steps of our proof.

4. We now turn to the *unitary groups* $\boldsymbol{U}_n(K_1, f_1)$ over quadratic number fields K_1. For an hermitian non-degenerate form f_1 in an n-dimensional space E_1 over the field K_1, there is a basis $(e_i)_{1 \leqq i \leqq n}$ of E_1 such that $f_1(z, z) = \sum_{k=1}^{n} \alpha_k \zeta_k \bar{\zeta}_k$, where the α_k are *rational* numbers; if $K_1 = \boldsymbol{Q}(\omega)$, where $\omega^2 = \rho$ is rational, and $\zeta_k = \xi_k + \omega\eta_k$, ξ_k and η_k rational, one has $f_1(z, z) = \sum_{k=1}^{n} \alpha_k(\xi_k^2 - \rho\eta_k^2)$. This, together with Meyer's theorem, shows at once that if $n \geqq 3$, and either $\rho > 0$ or the α_k have not all the same sign, then the form f_1 has an index $\geqq 1$; I have shown that in such a case, the projective unitary group $\boldsymbol{PU}^+{}_n(K_1, f_1)$ is simple [1, p. 70]. The only remaining case is therefore that in which K_1 is an imaginary quadratic field and the form f_1 is positive definite.

We shall always suppose, in what follows, that $n \geqq 4$ and the index ν of f_1 is 0. We begin by proving some preliminary lemmas.

LEMMA 1. *The commutator subgroup* Υ_n *of* $\boldsymbol{U}_n(K_1, f_1)$ *is identical with the commutator subgroup* Υ^+_n *of* $\boldsymbol{U}^+_n(K_1, f_1)$.

One has obviously $\Upsilon^+_n \subset \Upsilon_n$, and the only thing to prove is that every element in Υ_n belongs to Υ^+_n. Let us call *quasi-symmetries* the transformations in $\boldsymbol{U}_n$ which leave invariant all elements of a hyperplane in E_1; if $(e_i)_{1 \leqq i \leqq n}$ is an orthogonal basis of E_1 such that the e_i of index $\leqq n-1$ form a basis of a hyperplane H, e_n being orthogonal to H, any transformation w leaving invariant every element of H and belonging to $\boldsymbol{U}_n$, is such that $w(e_i) = e_i$ for $1 \leqq i \leqq n-1$ and $w(e_n) = \delta e_n$ with $\delta\bar{\delta} = 1$, and conversely. It follows immediately that every transformation in $\boldsymbol{U}_n$ can be written sv where $v \in \boldsymbol{U}^+_n$ and s is a quasi-symmetry of fixed hyperplane H; we are going to see that it is enough to prove that every commutator $svs^{-1}v^{-1}$ of a quasi-symmetry of hyperplane H and a transformation $v \in \boldsymbol{U}^+_n$ belongs to Υ^+_n. Indeed, let Γ be the subgroup of Υ_n generated by Υ^+_n and these commutators; Γ is a normal subgroup in $\boldsymbol{U}_n$, for one has, for a quasi-symmetry s_0 of hyperplane H,

$$s_0(svs^{-1}v^{-1})s_0^{-1} = (s_0 s)v(s_0 s)^{-1}v^{-1} \cdot vs_0v^{-1}s_0^{-1}$$

and for any $v_0 \in \boldsymbol{U}^+_n$

$$v_0(svs^{-1}v^{-1})v_0^{-1} = sus^{-1}u^{-1} \cdot s^{-1}v_0sv_0^{-1}$$

with $u = s^{-1}v_0sv \in \boldsymbol{U}^+_n$. In the quotient group $\boldsymbol{U}_n/\Gamma$, the classes of quasi-symmetries (of hyperplane H) commute therefore with all classes of elements of $\boldsymbol{U}^+_n$ but these latter commute among themselves, and so do the quasi-symmetries of hyperplane H; therefore, as $\boldsymbol{U}_n/\Gamma$ is generated by all these classes, it is an abelian group, and Γ contains Υ_n, which proves that it is identical to Υ_n. We thus have only to prove that $svs^{-1}v^{-1} \in \Upsilon^+_n$, or in other words, if s_1 and s_2 are two conjugate quasi-symmetries in $\boldsymbol{U}_n$, that $s_1s_2^{-1} \in \Upsilon^+_n$. Suppose s_1 (resp. s_2) is a quasi-symmetry of hyperplane H_1 (resp. H_2); by assumption, there is a vector a_1 orthogonal to H_1 and a vector a_2 orthogonal to H_2 such that $f_1(a_1, a_1) = f_1(a_2, a_2)$, and $s_1(a_1) = \alpha a_1$, $s_2(a_2) = \alpha a_2$, with $\alpha\alpha = 1$. The intersection $H_1 \cap H_2$ is at least 2-dimensional over K_1, hence at least 4-dimnsional over $\boldsymbol{Q}$; by Meyer's theorem, there exists in $H_1 \cap H_2$ a vector c such that $f_1(c, c) = f_1(a_1, a_1) = f_1(a_2, a_2)$. Let s_0 be the quasi-symmetry of hyperplane H_0 orthogonal to c, and such that $s_0(c) = \alpha c$; one can write $s_1s_2^{-1} = s_1s_0s_0^{-1}s_2^{-1} = s_1s_0(s_2s_0)^{-1}$. By Witt's theorem, there is a transformation $w \in \boldsymbol{U}_n$ such that $w(a_1) = a_2$, $w(c) = c$; if w has a determinant $\delta \neq 1$, multiplying w by a quasi-symmetry of determinants δ^{-1}, whose hyperplane contains a_2 and c, gives a transformation $v \in \boldsymbol{U}^+_n$ such that

$v(a_1) = a_2$, $v(c) = c$; therefore one has $s_2 s_0 = v(s_1 s_0) v^{-1}$; as $s_1 s_0$ belongs to $\boldsymbol{U}^+_n$, this ends the proof of Lemma 1.

5. LEMMA 2. *Let P and P' be two planes in E_1 such that the restrictions of f_1 to P and P' are equivalent. Then, if $n \geqq 5$, there is a transformation v in the commutator subgroup Υ_n such that $v(P) = P'$.*

This is of course a reinforcement of Witt's theorem, which merely asserts the existence of v in $\boldsymbol{U}_n$ having the required property. Let a, b be an orthogonal basis of P, and a', b' an orthogonal basis of P' such that $f_1(a', a') = f_1(a, a)$ and $f_1(b', b') = f_1(b, b)$. Let us prove first there is a $w_1 \in \Upsilon_n$ such that $w_1(a) = a'$. Indeed, the subspace M orthogonal to a and a' is at least 2-dimensional over K_1, hence at least 4-dimensional over $\boldsymbol{Q}$; by Meyer's theorem there is a vector $c \in M$ such that $f_1(c, c) = f_1(a, a) = f_1(a', a')$. As a and c are orthogonal, there is a symmetry s such that $s(a) = c$, namely the symmetry with respect to the hyperplane orthogonal to $d = c - a$; similarly, the symmetry s' with respect to the hyperplane orthogonal to $d' = c - a'$ is such that $s'(c) = a'$. Now, as $f_1(d, d) = f_1(d', d')$, it follows from Witt's theorem that s and s' are conjugate in $\boldsymbol{U}_n$; therefore $w_1 = s's = s's^{-1} \in \Upsilon_n$, and one has $w_1(a) = a'$.

We can now suppose that $a = a'$, and show the existence of a transformation $w_2 \in \Upsilon_n$ such that $w_2(a) = a$ and $w_2(b) = b'$. If H is the hyperplane orthogonal to a, H is at least 4-dimensional over K_1, and contains b and b'; the preceding argument may then be repeated in H instead of E_1, and this proves the lemma.

6. LEMMA 3. *For $n \geqq 2$ (and $\nu = 0$), the group $\boldsymbol{U}^+_n(K_1, f_1)$ is generated by plane rotations.*[2]

This is a simple consequence of Witt's theorem. Let us apply induction on n, the result being trivial for $n = 2$. Let u be any transformation in $\boldsymbol{U}^+_n$, x any vector $\neq 0$; if $u(x) = x$, the hyperplane H orthogonal to x is invariant

[2] The statement made (without proof) in [1, p. 66] to the effect that every transformation in $\boldsymbol{U}^{\pm}_n(K_1, f_1)$ is a product of at most n symmetries, *is not correct.* For instance, if $n = 2$, and a is an element in K such that $a^2 = -1$, $a\bar{a} = 1$, it is easy to verify that the transformation $x \to ax$ cannot be the product of 2 symmetries. I do not even know if in general (when $\nu = 0$) the group $\boldsymbol{U}^{\pm}_2$ is generated by symmetries. This of course is equivalent to the following question: when x and y are two vectors such that $f_1(x, x) = f_1(y, y)$, does there exist a product of symmetries sending x into y? Certainly one symmetry alone cannot do it, for the two vectors $x + y$ and $x - y$ are not orthogonal in general (in contrast to what happens for the orthogonal groups in similar circumstances).

by u, hence the restriction of u to H is a product of plane rotations by the inductive hypothesis, and so is u itself. If $u(x) \neq x$, let P be a plane containing x and $u(x)$; as $f_1(x, x) = f_1(u(x), u(x))$, there is a transformation $v \in \boldsymbol{U}_n$, leaving invariant every element of P^*, and such that $v(x) = u(x)$; if v has a determinant $\delta \neq 1$, let y be a vector in P orthogonal to x, and consider the quasi-symmetry s of hyperplane orthogonal to y and such that $s(y) = \delta^{-1}y$; $w = vs$ is then a plane rotation of plane P, such that $w(x) = u(x)$. Applying the inductive hypothesis to $w^{-1}u$ gives the result.

LEMMA 4. *For $n \geqq 5$, the commutator subgroup $\Upsilon_n(K_1, f_1)$ is generated by products of the type $v_1 v_2^{-1}$, where v_1 and v_2 are conjugate plane rotations, whose planes are both in a 4-dimensional subspace of E_1 such that the restriction of f_1 to that subspace has a square discriminant.*

It is easy to show that Υ_n is generated by products $u_1 w_2^{-1}$ where u_1 and u_2 are *arbitrary* conjugate plane rotations. Indeed, in any commutator $uwu^{-1}w^{-1}$ of elements of $\boldsymbol{U}^+_n$, u is a product of a certain number p of plane rotations (Lemma 3); the result being obvious for $p = 1$, one has only to argue inductively on p (see [1], p. 23); if $u = st$, where s is a plane rotation and t a product of $p - 1$ plane rotations, one has

$$uwu^{-1}w^{-1} = s(twt^{-1}w^{-1})s^{-1} \cdot sws^{-1}w^{-1}$$

which proves our assertion.

Let now w_1 and u_2 be conjugate plane rotations, and let M be a 5-dimensional subspace of E_1 containing the planes P_1, P_2 of u_1 and u_2. Let us remark that, given any positive rational numbers α, β, and any 3-dimensional subspace N of E_1, one can always find in N two orthogonal vectors a, b such that $f_1(a, a) = \alpha$, $f_1(b, b) = \beta$, by two successive applications of Meyer's theorem. Therefore, we can first find in $P^*_1 \cap M$ a plane Q_1 such that the restriction of f_1 to $N_1 = P_1 + Q_1$ has a square discriminant, and similarly we can find a plane $Q_2 \subset P^*_2 \cap M$ such that the restriction of f_1 to $N_2 = P_2 + Q_2$ has a square discriminant. Now, as N_1 and N_2 are both in M and are 4-dimensional, their intersection N is at least 3-dimensional; therefore, there exists a plane $P_3 \subset N$ such that the restriction of f_1 to P_3 is equivalent to its restrictions to P_1 and P_2. There is therefore, by Witt's theorem, a plane rotation u_3 having P_3 as its plane, and conjugate to both u_1 and u_2 in $\boldsymbol{U}_n$; as $w_1 u_2^{-1} = (u_1 u_3^{-1})(u_3 u_2^{-1})$, our lemma is proved.

7. LEMMA 5. *When $n = 4$ and f_1 has a square discriminant, the group $P\Upsilon_4(K_1, f_1)$ is simple, and Υ_4 is generated by the squares of plane rotations.*

We can always multiply f_1 by a positive rational number α, and choose a basis in E_1 such that $g_1 = \alpha f_1$ is given by the expression

$$g_1(z, z) = \zeta_1\bar{\zeta}_1 + a_2a_3\zeta_2\bar{\zeta}_2 + a_3a_1\zeta_3\bar{\zeta}_3 + a_1a_2\zeta_4\bar{\zeta}_4,$$

where a_1, a_2, a_3 are positive rational numbers. The well-known isomorphism between orthogonal groups in 6 variables and unitary groups in 4 variables ([4, p. 24] and [3, n° 33]) is such that g_1 corresponds to the positive definite symmetric bilinear form $g(x, y)$ in 6 variables over $\boldsymbol{Q}$ defined by

$$g(x, x) = a_1\xi_1^2 - \rho a_1\xi_2^2 + a_2\xi_3^2 - \rho a_2\xi_4^2 + a_3\xi_5^2 - \rho a_3\xi_6^2$$

in the following way; there exists an homomorphism of $\boldsymbol{U}^+_4(K_1, g_1)$ onto a subgroup Γ of $\boldsymbol{O}^+_6(\boldsymbol{Q}, g)$ such that $\boldsymbol{O}^+_6/\Gamma$ is abelian (which means that $\Omega_6 \subset \Gamma$), the kernel Θ of that homomorphism consisting of the identity and the transformation $z \to -z$. The subgroup Δ of $\boldsymbol{U}^+_4(K_1, g_1)$ generated by the squares of the elements of $\boldsymbol{U}^+_4$ (or the group Δ/Θ if $\Theta \subset \Delta$) is therefore isomorphic to the group generated by the squares of the elements of Γ, hence to the commutator subgroup $\Omega_6(\boldsymbol{Q}, g)$, since that group contains the squares of the elements of $\boldsymbol{O}^+_6$ [1, pp. 23-24] and is simple over its center (n° 3). Therefore Δ is also simple over its center, and as it contains Υ_4 (since $\boldsymbol{U}^+_4/\Delta$ is abelian), Υ_4 is simple over its center and identical to Δ. Moreover, Υ_4 contains the squares of all plane rotations, hence is generated by them because of the simplicity of $P\Upsilon_4$.

8. We can now proceed to the proof of our main

THEOREM. *For any positive definite hermitian form f_1 in $n \geqq 5$ variables over an imaginary quadratic extension K_1 of the rational field, the factor group $P\Upsilon_n(K_1, f_1)$ of $\Upsilon_n(K_1, f_1)$ by its center $\Upsilon_n \cap Z_n$ is simple.*

We first remark that the square of every plane rotation belongs to Υ_n. Indeed, let P be the plane of a rotation v; by Meyer's theorem, there is a 4-dimensional subspace L containing P and such that the restriction f_0 of f_1 to L has a square discriminant. But the subgroup of $\boldsymbol{U}^+_n(K_1, f_1)$ leaving every element of L^* invariant is isomorphic to (and can be identified with) the group $\boldsymbol{U}^+_4(K_1, f_0)$; hence Lemma 5 shows that the square of v belongs to commutator subgroup of $\boldsymbol{U}^+_4(K_1, f_0)$, and a fortiori to Υ_n.

Let now G be a normal subgroup in Υ_n, not contained in the center. Our first step consists in showing that if $u \in G$ is not in the center, there exists a plane $P \subset E_1$ such that $u(P) \neq P$ and $M = P + u(P)$ is 3-dimensional. Owing to the initial remark, this can be done by exactly the same

argument as in [2, n° 2], which we therefore suppress here (the only difference consisting in replacing the bilinear form $f(x, y)$ by $\frac{1}{2}(f_1(x, y) + f_1(y, x))$ in the calculation).

As the space M^* is at least 2-dimensional, there exists, by Meyer's theorem, a vector $c \in M^*$ such that the restriction f_2 of f_1 to the 4-dimensional subspace F generated by M and c has a square discriminant. Let v_1 be a plane rotation of plane P, such that $v = v_1^2$ is not the identity; v belongs to Υ_n and to the commutator subgroup of $\boldsymbol{U}^+{}_4(K_1, f_2)$. It follows that $u_1 = v^{-1}uvu^{-1}$ belongs to G, is not in the center, and leaves invariant every element of M^*; moreover, as uvu^{-1} is the square of a plane rotation of plane $u(P) \subset F$, it belongs also to the commutator subgroup of $\boldsymbol{U}^+{}_4(K_1, f_2)$, and so does u_1. Lemma 5 then proves that G contains every element in that commutator subgroup, and in particular G contains *every square of a plane rotation* whose plane is contained in F. But if w is the square of *any* plane rotation of plane Q, there exists a transformation $s \in \Upsilon_n$ such that sws^{-1} has its plane contained in F: for Meyer's theorem proves the existence in F of a plane Q' such that the restriction of f_1 to Q' is equivalent to the restriction of f_1 to Q, and then Lemma 2 provides a transformation $s \in \Upsilon_n$ such that $s(Q) = Q'$. Therefore, G contains the square of *every* plane rotation.

Now let t_1 and t_2 be two conjugate plane rotations whose planes are contained in a 4-dimensional subspace N of E_1 such that the restriction f_3 of f to N has a square discriminant; Lemma 5 shows that the commutator subgroup of $\boldsymbol{U}^+{}_4(K_1, f_3)$ is simple over its center. Moreover, if w is the square of a plane rotation whose plane is in N, w belongs to the commutator subgroup of $\boldsymbol{U}^+{}_4(K_1, f_3)$. But we have seen that w belongs to G, hence it follows from Lemma 5 that G contains the commutator subgroup of $\boldsymbol{U}^+{}_4(K_1, f_3)$, and in particular it contains $t_1t_2^{-1}$ (for the restrictions of t_1 and t_2 to N are conjugate *within the group* $\boldsymbol{U}_4(K_1, f_3)$ by Witt's theorem, the restrictions of f_3 to their planes being equivalent). Lemma 4 proves finally that G is identical to Υ_n, and this completes the proof of the theorem.

As a corollary, it follows from that proof that Υ_n is *generated by the squares of the rotations in* $\boldsymbol{U}^+{}_n$.

The theorem (and its proof) are of course valid for any subfield of $\boldsymbol{R}$ in which Meyer's theorem is true.

9. When a hermitian form f_1 has an index $\nu \geqq 1$, it is known [1, p. 70] that, for any field K_1 and any dimension $n \geqq 2$ (with the exceptions $n = 2$, $K_1 = \boldsymbol{F}_4$ or $K_1 = \boldsymbol{F}_9$, and $n = 3$, $K_1 = \boldsymbol{F}_4$), the commutator subgroup $\Upsilon^+{}_n$ of $\boldsymbol{U}^+{}_n$ is *identical with* $\boldsymbol{U}^+{}_n$. That this is *not* always the case when the index

$\nu = 0$ can be seen on the following example: consider, over the quadratic field $K_1 = \boldsymbol{Q}(2^{\frac{1}{2}})$, the hermitian form in 2 variables $f_1(z, z) = \zeta_1\bar{\zeta}_1 + 5\zeta_2\bar{\zeta}_2$. The unitary group $\boldsymbol{U}^+{}_2(K_1, f_1)$ is then isomorphic to a subgroup Γ of the orthogonal group $\boldsymbol{O}^+{}_4(\boldsymbol{Q}, f)$ relative to the form $f(x, x) = \xi_1{}^2 + 2\xi_2{}^2 + 5(\xi_3{}^2 + 2\xi_4{}^2)$, the isomorphism being defined in the well-known fashion which, to every unitary matrix

$$\begin{pmatrix} \alpha & 5\beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

in $\boldsymbol{U}^+{}_2(K_1, f_1)$ (with $\alpha\bar{\alpha} + 5\beta\bar{\beta} = 1$) associates the orthogonal matrix of the corresponding linear substitution on the ξ_k, related to ζ_1 and ζ_2 by $\zeta_1 = \xi_1 + \omega\xi_2$ and $\zeta_2 = \xi_3 + \omega\xi_4$, with $\omega^2 = -2$. Now, to the transformation $z \to -z$, which is in $\boldsymbol{U}^+{}_2(K_1, f_1)$, is thus associated the symmetry $x \to -x$ in Γ; but I have proved [2, n° 11], that that transformation does not belong to the commutator subgroup $\Omega_4(\boldsymbol{Q}, f)$, and a fortiori it cannot belong to the commutator subgroup of Γ; hence $\boldsymbol{U}^+{}_2(K_1, f_1)$ is in that case distinct from its commutator subgroup. It would be interesting to know if, for $n \geqq 5$, $\nu = 0$ and K_1 an imaginary quadratic extension of $\boldsymbol{Q}$, the commutator subgroup Υ_n of $\boldsymbol{U}^+{}_n(K_1, f_1)$ is or is not identical to $\boldsymbol{U}^+{}_n$. I have not been able to find an answer to this question.

THE JOHNS HOPKINS UNIVERSITY.

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ON GEODESIC COORDINATES.*

By PHILIP HARTMAN.**

1. The problem. Let $S: X = X(u, v)$ be a surface of class C''. That is, let $X = (x, y, z)$ be a vector in 3-space, and $X = X(u, v)$ a function of class C'' in a vicinity of $(u, v) = (0, 0)$ such that the vector product (X_u, X_v) is not zero (the subscripts u, v denote partial differentiation). Let the squared element of arc-length on S be

$$ds^2 = g_{ik}(u, v)\, du^i du^k, \qquad (u^1, u^2) = (u, v), \tag{1}$$

where, without loss of generality, it can be supposed that

$$g_{11}(0, 0) = 1, \qquad g_{12}(0, 0) = 0, \qquad g_{22}(0, 0) = 1. \tag{2}$$

The functions g_{11}, g_{12}, g_{22}, which are the scalar products $X_u \cdot X_u$, $X_u \cdot X_v$, $X_v \cdot X_v$, respectively, are of class C'.

It has recently been shown [1] that it is possible to introduce geodesic polar coordinates (r, ϕ) of class C' at the point $(0, 0)$. By this is meant that there exist functions $u = u(r, \phi)$, $v = v(r, \phi)$ of class C' on a set $0 \leqq r \leqq$ const., $-\infty < \phi < \infty$, with the properties that, for a fixed ϕ, the arc $u = u(r, \phi)$, $v = v(r, \phi)$ is the unique geodesic satisfying the initial conditions

$$u(0, \phi) = 0,\ v(0, \phi) = 0 \text{ and } u_r(0, \phi) = \cos\phi,\ v_r(0, \phi) = \sin\phi; \tag{3}$$

r is the arc-length on this geodesic; the Jacobian $\partial(u, v)/\partial(r, \phi)$ does not vanish for $r \neq 0$; finally, the transformation $(u, v) \to (r\cos\phi, r\sin\phi)$ is one-to-one. It was also shown that the functions $u_r(r, \phi)$, $v_r(r, \phi)$ are of class C'. It was left an open question whether or not the transformation $(u, v) \to (r, \phi)$ is of class C'' (that is, whether or not the partial derivatives $u_{\phi\phi}$, $v_{\phi\phi}$ exist and are continuous; although it was pointed out that, for a fixed r, the arc $u = u(r, \phi)$, $v = v(r, \phi)$ is of class C'' in terms of its arc-length parameter).

The object of this paper is to give an example showing that the answer to the question can be in the negative. The example S, to be given below, is curious for additional reasons. First, the coefficient functions g_{ik} in (1) are of class C''; cf. (4) and (5) below. Of course, S cannot be of class C'''

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** John Simon Guggenheim Memorial Foundation Fellow, on leave of absence from The Johns Hopkins University.

(by the remark preceding the statement of Theorem 2 in [1], p. 724), nor can the Gaussian curvature of S be of class C' (by § 6 in [1] and the Lemma, § 12, [4]).

Another curious property of S is that, at the point $(u, v) = (0, 0)$, there exists one choice of geodesic parallel coordinates which is not of class C'', while all other choices are of class C''.

The example S turns out to be a surface of revolution and is only a slight modification of the one given in [1], p. 724, to show that if the functions g_{ik} in the Riemannian metric (1) are of class C' (but (1) does not belong to a surface of class C'', nor does (1) possess a bounded Gaussian curvature; cf. Theorem (III) in [3], p. 133), then the geodesics need not be uniquely determined by initial conditions; cf. also the example in [2], p. 554 and the Remark in [3], p. 138.

2. The example. Let $h(v) > 0$ be defined for small $|v|$ by

$$h(v) = 1 + |v|^{\lambda}, \text{ where } 2 < \lambda < 3; \tag{4}$$

so that

$$h(0) = 1 \text{ and } h_v(0) = 0. \tag{5}$$

It is clear that $h(v)$ is of class C'' and, in fact, its second derivative satisfies a uniform Hölder condition of order $\lambda - 2$ near $v = 0$. Consider the case

$$ds^2 = h(v)(du^2 + dv^2) \tag{6}$$

of (1). That (6) is the first fundamental form on a surface S of class C'' follows from the Remark [3], p. 138. In fact, such an S is given by $S\colon X = (f(v)\cos u, f(v)\sin u, g(v))$, where

$$f(v) = h^{\frac{1}{2}}(v) \text{ and } g(v) = \int_0^v (h - h_v^2/4h^2)^{\frac{1}{2}} dt$$

(the reality of f and g for small $|v|$ follows from (5)).

The differential equations of the geodesics, belonging to (6), are

$$(h(v)u')' = 0, \qquad (h(v)v')' - \tfrac{1}{2}h_v(v)(u'^2 + v'^2) = 0, \tag{7}$$

where the prime denotes differentiation with respect to a variable which does not occur explicitly and which can be assumed to be the arc-length. In this case, one has the two first integrals

$$h(v)u' = \text{const.}, \qquad h(v)(u'^2 + v'^2) = 1, \tag{8}$$

by (6).

3. Geodesic polar coordinates. In the case of the intial conditions (3), the first part of (5) shows that if $\phi \neq 0$ is fixed, the equations (8) become

$$(9) \qquad h(v)\,du/\cos\phi = dr, \qquad (\operatorname{sgn}\phi)h(v)\,(h(v) - \cos^2\phi)^{-\frac{1}{2}}\,dv = dr;$$

and that if $\phi = 0$, the equations (8) have the solution

$$(10) \qquad u = u(r, 0) = r, \qquad v = v(r, 0) = 0.$$

In order to avoid absolute value signs in what follows, it will be assumed that $\phi \geqq 0$, $r \geqq 0$, hence $v \geqq 0$. The definition (4) of h and the second part of (9) give

$$(11) \qquad \int_0^v (1 + t^\lambda)\,(t^\lambda + \sin^2\phi)^{-\frac{1}{2}}\,dt = r, \text{ where } v = v(r, \phi).$$

It is clear from (10) that $v(r, \phi) = o(1)$, as $\phi \to 0$, uniformly in r, say for $0 \leqq r \leqq \text{const}$. Thus (11) implies that there exists a number $\theta = \theta(r, \phi)$, where $0 < \theta < 1$, satisfying $v(1 + o(1))\,(\theta v^\lambda + \sin^2\phi)^{-\frac{1}{2}} = r$. This relation can be written as $v^2(1 + o(1) - \theta r^2 v^{\lambda-2}) = r^2 \sin^2\phi$. Hence, uniformly in r,

$$(12) \qquad v = r(1 + o(1))\sin\phi, \text{ as } \phi \to 0.$$

If one introduces $t/\sin\phi$ as a new integration variable, (11) becomes

$$\int_0^{v/\sin\phi} (1 + t^\lambda \sin^\lambda\phi)\,(1 + t^\lambda \sin^{\lambda-2}\phi)^{-\frac{1}{2}}\,dt = r.$$

By (12), the second factor of the integrand can be written as

$$1 - \tfrac{1}{2}t^\lambda(1 + o(1))\sin^{\lambda-2}\phi;$$

hence,

$$\int_0^{v/\sin\phi} (1 - \tfrac{1}{2}t^\lambda(1 + o(1))\sin^{\lambda-2}\phi)\,dt = r.$$

Thus $v/\sin\phi - \frac{1}{2}(\lambda + 1)^{-1}v^{1+\lambda}(1 + o(1))/\sin^3\phi = r$ or, by (12),

$$v = r\sin\phi + \tfrac{1}{2}(\lambda + 1)^{-1}r^{1+\lambda}(1 + o(1))\sin^{\lambda-1}\phi.$$

Since $\lambda - 1 < 2$ and $\sin\phi = \phi + O(\phi^2)$,

$$v(r, \phi) = r\sin\phi + \tfrac{1}{2}(1 + \lambda)^{-1}r^{1+\lambda}\phi^{\lambda-1}(1 + o(1)), \text{ as } \phi \to 0,$$

uniformly in r. The last formula line shows that $v_\phi(r, 0) = r$, but since $\lambda - 1 < 2$, the partial derivative $v_{\phi\phi}(r, 0)$ fails to exist for $r > 0$. In other words, the transformation $(u, v) \to (r, \phi)$ is not of class C''.

4. Geodesic parallel coordinates. Let ϕ be fixed and choose as the base geodesic K_ϕ: $u = u(\tau) = u(\tau, \phi)$, $v = v(\tau) = v(\tau, \phi)$, where K_ϕ is defined by (9) or (10) according as $0 < \phi < \pi$ or $\phi = 0$. Let σ denote arc-length along the geodesic $u = U(\sigma, \tau)$, $v = V(\sigma, \tau)$ which, for $\sigma = 0$, is orthogonal to K_ϕ at the point $\tau = \tau$. Thus σ, τ are geodesic parallel coordinates.

It can be supposed that U, V satisfy the respective initial conditions $U(0, \tau) = u(\tau)$, $V(0, \tau) = v(\tau)$ and $U_\sigma(0, \tau) = -v_\tau(\tau)$, $V_\sigma(0, \tau) = u_\tau(\tau)$. Then (8) becomes the system of differential equations

$$dU/d\sigma = -h(v(\tau))v_\tau(\tau)/h(V), \quad dV/d\sigma = (h(V) - h^2(v(\tau))v_\tau^2(\tau))^{\frac{1}{2}}/h(V).$$

Since $V(0, 0) = v(0) = 0$ and $v_\tau(0) = \sin\phi$, it follows that, at $(\sigma, \tau) = (0, 0)$, the square-root occurring on the right-hand side of the last differential equation is $\cos\phi$, which is not 0 if $\phi \neq \frac{1}{2}\pi$. Thus, if $\phi \neq \frac{1}{2}\pi$, the functions of U, V, τ on the right-hand sides of the differential equations are of class C'' for small $|U|$, $|V|$, $|\tau|$, since $v_\tau(\tau)$ is of class C'' by (9) or (10). Consequently, $V(\sigma, \tau)$, $V(\sigma, \tau)$ are of class C'' for small $|\sigma|$, $|\tau|$. In other words, the transformation $(u, v) \rightarrow (\sigma, \tau)$ to geodesic parallel coordinates is of class C'' when the base geodesic is K_ϕ, where $\phi \neq \frac{1}{2}\pi$.

On the other hand, it will be shown that if the geodesic $K_{\frac{1}{2}\pi}$,

$$(13) \qquad u \equiv 0, \; v = v(\tau), \text{ where } \int_0^v h^{\frac{1}{2}}(t)\,dt = \tau,$$

is chosen as the base geodesic, the resulting transformation $(u, v) \rightarrow (\sigma, \tau)$ to geodesic parallel coordinates is not of class C''.

Let $u = U(\sigma, \tau)$, $v = V(\sigma, \tau)$ be the geodesic orthogonal to (13) at the point $\tau = \tau$, when the arc-length parameter σ is 0. As noted in [1], the methods there imply that the functions $U(\sigma, \tau)$, $V(\sigma, \tau)$ are of class C' on some set $|\sigma| \leqq \text{const.}$, $|\tau| \leqq \text{Const.}$ The assertion to be proved is that they are not of class C''.

In what follows, the letter v denotes $v(\tau)$, the function in (13) at $\tau = \tau$; the letter V, the function $V(\sigma, \tau)$. In order to avoid absolute value signs, it will be supposed that $\sigma \geqq 0$, $\tau \geqq 0$, hence $v \geqq 0$, $V \geqq 0$. All square roots will be non-negative.

The functions U, V satisfy the respective initial condition $U(0, \tau) = 0$, $V(0, \tau) = v(\tau)$ and $U_\sigma(0, \tau) = h^{-\frac{1}{2}}(v)$, $V_\sigma(0, \tau) = 0$. Thus, if $\tau \neq 0$ is fixed, (8) becomes

$$(14) \qquad h(V)\,dU = h^{\frac{1}{2}}(v)\,d\sigma, \qquad h(V)\,(h(V) - h(v))^{-\frac{1}{2}}\,dV = d\sigma;$$

while if $\tau = 0$, (8) has the solution

$$U(\sigma, 0) = \sigma, \qquad V(\sigma, 0) = 0. \tag{15}$$

Consequently, (4) and an integration of the second relation in (14) give

$$\int_v^V (1 + t^\lambda)(t^\lambda - v^\lambda)^{-\frac{1}{2}}\, dt = \sigma. \tag{16}$$

If $(t^\lambda - v^\lambda)^{\frac{1}{2}}$ is introduced as a new integration variable, then (16) becomes

$$\int_0^L (1 + t^2 + v^\lambda)(t^2 + v^\lambda)^{-1+1/\lambda}\, dt = \tfrac{1}{2}\lambda\sigma, \tag{17}$$

where

$$L = (V^\lambda - v^\lambda)^{\frac{1}{2}}. \tag{18}$$

The second relation in (15) implies that

$$V(\sigma, \tau) = \tau(V_\tau(\sigma, 0) + o(1)), \text{ as } \tau \to 0, \tag{19}$$

uniformly in σ, where $0 \leqq \sigma \leqq$ const.; in particular, if $\sigma = 0$, then by (13) and (4),

$$v(\tau) = \tau(1 + o(1)), \text{ where } V_\tau(0, 0) = 1. \tag{20}$$

As t varies from 0 to L, the expression $t^2 + v^\lambda$ in (17) varies from $v^\lambda = V^\lambda(0, \tau)$ to $V^\lambda(\sigma, \tau)$. Thus there exists a $\theta_1 = \theta_1(\sigma, \tau)$, satisfying $0 < \theta_1 < 1$ and

$$L(1 + o(1))\, V^{1-\lambda}(\theta_1\sigma, \tau) = \tfrac{1}{2}\lambda\sigma.$$

By (19), the last relation is equivalent to

$$L = \tfrac{1}{2}\lambda\sigma\tau^{\lambda-1}(V_\tau{}^{\lambda-1}(\theta_1\sigma, 0) + o(1)).$$

The definition (18) of L and the mean-value theorem of differential calculus give $L^2 = \lambda(V - \sigma)m^{\lambda-1}$ for some number m satisfying $v < m < V$. In view of (19), this m is of the form $\tau(V_\tau(\theta_2\sigma, 0) + o(1))$, where $\theta_2 = \theta_2(\sigma, \tau)$ satisfies $0 < \theta_2 < 1$. Consequently, by the last formula line, $V(\sigma, \tau)$ is identical with

$$v(\tau) + (\lambda/4)\sigma^2\tau^{\lambda-1}V_\tau{}^{2\lambda-2}(\theta_1\sigma, 0)\, V_\tau{}^{1-\lambda}(\theta_2\sigma, 0)\,(1 + o(1)).$$

If this is divided by τ, and $\tau \to 0$, it is seen that $V_\tau(\sigma, 0) \equiv 1$. This is a consequences of (20) and the fact that $\lambda - 1 > 1$, while $V_\tau(\sigma, 0)$ keeps away

from 0 for all sufficiently small $|\sigma|$. Thus the representation of $V(\sigma, \tau)$ by the last formula line simplifies to

$$V(\sigma, \tau) = v(\tau) + (\lambda/4)\sigma^2\tau^{\lambda-1}(1 + o(1)).$$

Since $\lambda - 1 < 2$ and $v(\tau) = \tau + O(\tau^2)$, it follows that

$$V(\sigma, \tau) = \tau + (\lambda/4)\sigma^2\tau^{\lambda-1}(1 + o(1)), \text{ as } \tau \to 0.$$

This equation and the inequality $\lambda - 1 < 2$ make is clear that $V_{\tau\tau}$ fails to exist at any point $(\sigma, 0)$ where $\sigma \neq 0$. This completes the proof.

PARIS, FRANCE.

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ON LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH SMALL COEFFICIENTS.*

By PHILIP HARTMAN.**

Let $f(t)$ be a (real-valued) continuous function for large t-values, say for $T \leqq t < \infty$. The differential equation

$$x'' + fx = 0 \tag{1}$$

is said to be oscillatory or non-oscillatory according as one (hence every) solution $x = x(t) \not\equiv 0$ of (1) has or does not have an infinity of zeros on $T \leqq t < \infty$. Since

$$x'' + cx/t^2 = 0 \tag{2}$$

is oscillatory or non-oscillatory on $1 \leqq t < \infty$ according as $c > \frac{1}{4}$ or $c \leqq \frac{1}{4}$ (Kneser), the condition

$$F(t) = \lim_{X \to \infty} \int_t^X f(s)\,ds = \int_t^\infty f(s)\,ds \text{ is convergent} \tag{3}$$

is compatible with (1) being either oscillatory or non-oscillatory.

The following result was proved in [2], § 6:

(*) If (1) is non-oscillatory and (3) holds, then, for any solution $x = x(t) \not\equiv 0$ of (1), the logarithmic derivative $l = x'/x$ (which exists for large t-values) satisfies the Riccati differential equation

$$l' + l^2 + f = 0 \tag{4}$$

and the relations

$$\int^\infty l^2 dt < \infty \tag{5}$$

and

$$l(t) = F(t) + L(t), \tag{6}$$

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** John Simon Guggenheim Memorial Foundation Fellow, on leave of absence from The Johns Hopkins University.

where $F(t)$ is given by (3), and $L(t)$ by

$$L(t) = \int_t^\infty l^2(s)\,ds. \tag{7}$$

With the aid of (*), it was shown in [2], § 7, that if (3) holds and

$$\int_T^\infty \exp(-A\int_T^t F(s)\,ds)\,dt < \infty, \tag{8}$$

where $A = 2$, then (1) is oscillatory. It was pointed out that, in view of the cases $c > \frac{1}{4}$ of (2), it is not possible to replace $A = 2$ by any number A exceeding 4, and the question was raised as to whether or not $A = 2$ can be replaced by $A = 4$. It will be shown in (I) below that the answer to this question is in the affirmative.

The criterion (8) for an oscillatory equation (1) is complementary to the following criterion of [2], § 2, for a non-oscillatory equation (1) (or even for the non-existence of any solution $x = x(t) \not\equiv 0$ with two zeros on $T \leqq t < \infty$):

$$-\tfrac{3}{4} \leqq tF(t) \leqq \tfrac{1}{4} \text{ for } t \geqq T, \qquad (T > 0), \tag{9}$$

where (3) is assumed. If, in the proof given in [2], § 3, for the non-oscillatory character of (1) in the case (9), the function $y = F(t) + 1/(4t)$ is replaced by $y = F(t) + \gamma^2/t$, where γ is a fixed number in the range $\frac{1}{2} \leqq \gamma \leqq 1$, it is seen that (9) can be replaced by

$$-\gamma - \gamma^2 \leqq tF(t) \leqq \gamma - \gamma^2 \text{ for } t \geqq T. \tag{10}$$

The criteria (10), belonging to different values of γ, are all distinct. But in a certain sense, the case $\gamma = \frac{1}{2}$ of (9) is "special." First, the $\frac{1}{4}$ on the right-hand side of (9) is the maximum value of $\gamma - \gamma^2$ for $\frac{1}{2} \leqq \gamma \leqq 1$ and cannot be replaced by a larger number. in view of the examples (2). On the other hand, as will be shown in (III) below, (9) implies that no solution of (1) satisfies

$$0 < \int_T^\infty x^2(t)\,dt < \infty, \tag{11}$$

and, in this connection, the $-\frac{3}{4}$ on the left-hand side of (9) cannot be replaced by a smaller number, in view of the examples (2). This statement has applications in the study of self-adjoint boundary problems on

$L_2(T,\infty)$ associated with (1). (It will be seen that the proof of (III) below can be modified to show that (10) implies that no solution of (1) satisfies

$$0 < \int_T^\infty x^{1/\gamma}(t)\,dt < \infty, \tag{11 bis}$$

and, in this connection, the number $-\gamma-\gamma^2$ on the left-hand side of (10) cannot be replaced by a smaller number; cf. the Remark following the proof of (III)).

With the aid of (*), it was shown in [2], § 8, that if (3) holds and (1) is non-oscillatory, then

$$\int_T^\infty \exp(B\int_T^t F(s)\,ds)\,dt = \infty, \tag{12}$$

where $B=2$, implies that no solution of (1) satisfies (11). It was pointed out that, in view of the cases $c<-\frac{3}{4}$ of (1), it is not possible to replace B by a number less than $\frac{4}{3}$, and the question was raised as to whether or not $B=2$ can be replaced by $B=\frac{4}{3}$; cf. also p. 370 of [2]. It will be shown in (III) below that the answer to this question is in the affirmative. (In particular, this implies the remark of the preceding paragraph concerning (11).)

(I) *Let $f(t)$, where $T\leqq t<\infty$, be a continuous function satisfying* (3). *Then* (8), *where $A=4$, implies that* (1) *is oscillatory. This assertion is false if $A>4$.*

It may be remarked that if (8) holds for some A, where $0<A<4$, then it holds for $A=4$. In order to see this, let $G(t)=\exp(-\int_T^t F(s)ds)$. Then (8) means that the A-th power, $G^A(t)$, is of class $L_1(T,\infty)$. It follows from the definition of G that $G'/G=-F=o(1)$, as $t\to\infty$. Hence, $(G^A)'=AG^{A-1}G'=AG^A(G'/G)$ is $AG^Ao(1)$; so that $(G^A)'$ is of class $L_1(T,\infty)$. Consequently $\lim G^A(t)$, as $t\to\infty$, exists and is 0; that is, $G(t)=o(1)$. Finally, $G^4=G^AG^{4-A}=G^Ao(1)$ is of class $L_1(T,\infty)$, which proves the remark.

Proof of (I). Suppose that the positive assertion in (I) is false. Then any solution $x=x(t)\not\equiv 0$ of (1) has only a finite number of zeros. It can

be supposed that T is chosen so large that a given solution $x = x(t)$ does not vanish for $t \geqq T$ (and that $T > 1$); the change of the lower limits of integration on the left-hand side of (8) clearly does not affect (8). It can also be supposed that $x(t) > 0$ for $t \geqq T$ (for otherwise x can be replaced by $-x$). Thus, if $l = x'/x$, then (4), (5) and (6) follow from (*).

The relations (4) and (6) imply that $L(t)$ satisfies the differential equation

$$L' + F^2 + 2FL + L^2 = 0; \tag{13}$$

and so,

$$L' + 4FL \leqq 0, \text{ that is, } (L(t) \exp (4 \int_T^t F(s)ds))' \leqq 0 \tag{14}$$

for $t \geqq T$. Since $L(t) \geqq 0$, by (7), it follows that

$$0 \leqq L(t) \leqq L(T) \exp (-4 \int_T^t F(s)ds) \text{ for } t \geqq T. \tag{15}$$

It follows from the assumptions of (I) that $\int_T^\infty L(t)dt < \infty$. The definition (7) of $L(t)$ shows that this is equivalent to

$$\int_T^\infty sl^2(s)ds < \infty. \tag{16}$$

Since $l(t)$ is the logarithmic derivative of $x(t)$,

$$\log(x(t)/x(T)) = \int_T^t l(s)ds. \tag{17}$$

If $l(s)$ is written as the product of $s^{\frac{1}{2}}l(s)$ and $s^{-\frac{1}{2}}$, Schwarz's inequality shows that

$$(\int_T^t l(s)ds)^2 \leqq (\int_T^t sl^2(s)ds)(\int_T^t s^{-1}ds). \tag{18}$$

Thus, by (16) and (17), $\log (x(t)/x(T)) \leqq C(\log t)^{\frac{1}{2}}$ for $t \geqq T(> 1)$ if C^2 denotes the value of the integral in (16). Hence $x(t) \leqq x(T) \exp C(\log t)^{\frac{1}{2}}$, which implies

$$\int_T^\infty x^{-2}(t)dt \geqq x^{-2}(T) \int_T^\infty \exp (-2C(\log t)^{\frac{1}{2}})dt = \infty.$$

Since $x = x(t) \not\equiv 0$ is an arbitrary (non-trivial) solution of (1), this leads to a contradiction. For, according to [1], p. 703, every non-oscillatory equation (1) has a solution $x = x(t) \not\equiv 0$ satisfying $\int^{\infty} x^{-2}(t)\,dt < \infty$. This proves the positive assertion in (I).

Since the negative assertion follows from the examples (2), the proof of (I) is complete.

The inequality (14), deduced from (13), has a dual, namely,

$$L' + 2(F^2 + L^2) \geqq 0. \tag{19}$$

It will be shown that this inequality and the main theorem of [2], § 1, lead to the following criterion that (1) be non-oscillatory:

(II) *Let $f(t)$, where $T \leqq t < \infty$, be a continuous function satisfying* (3). *If the differential equation*

$$z'' + 4F^2 z = 0 \tag{20}$$

is non-oscillatory, then (1) *is.*

It may be remarked that (II) is not a consequence of the Sturm comparison theorem. For example, if $f(t) = c/t^2$, as in (2), then $4F^2 = 4c^2/t^2$. Thus (20) is non-oscillatory if and only if $|\,c\,| \leqq \frac{1}{4}$, but $f(t) \leqq 4F^2$ if and only if either $c \leqq 0$ or $c \geqq \frac{1}{4}$.

Proof of (II). It can be supposed that T is chosen so large that (20) has a solution $z = z(t) > 0$ for $T \leqq t < \infty$. If $h = z'/z$ is the logarithmic derivative of z, then h satisfies the Riccati differential equation $h' + h^2 + 4F^2 = 0$; cf. (4). Let $g = \frac{1}{2}h$. Then $g' + 2(g^2 + F^2) = 0$; cf. (19).

Define $y = y(t)$ for $t \geqq T$ by placing $y = F(t) + g(t)$. Then $y(t)$ has a continuous derivative and satisfies the differential inequality $y' + y^2 + f = -(F - g)^2 \leqq 0$. According to [2], § 1, the existence of such a function $y = y(t)$ implies that a solution $x = x(t) \not\equiv 0$ cannot vanish twice for $T \leqq t < \infty$. This proves (II).

(III) *Let $f(t)$, where $T \leqq t < \infty$, be a continuous function satisfying* (3) *and, in addition, let* (1) *be non-oscillatory. Then, if* (12) *holds for $B = \frac{4}{3}$, no solution $x = x(t)$ of* (1) *satisfies* (11). *This assertion becomes false if $B = \frac{4}{3}$ is replaced by any $B < \frac{4}{3}$.*

If (12) holds for some $B > \frac{4}{3}$, then it holds for $B = \frac{4}{3}$. This can be verified as in the remark following the statement of (I).

Proof of (III). Suppose that the first assertion in (III) is false. Then (1) possesses a solution $x = x(t) \not\equiv 0$ which has only a finite number of zeros and satisfies (11). It can be supposed that the origin on the t-axis has been so chosen that $x(t) \neq 0$ for $0 \leqq t < \infty$, and that $x(t)$ has been normalized by

(21) $$x(0) = 1, \text{ so that } x(t) > 0 \text{ for } 0 \leqq t < \infty.$$

If $l = x'/x$, a quadrature of (6) gives

(22) $$\log x(t) = \int_0^t F(s)\,ds + \int_0^t L(s)\,ds.$$

But $\int_0^t L(s)\,ds = \int_0^t sl^2(s)\,ds + t\int_t^\infty l^2(s)\,ds$, by (7); hence

(23) $$\int_0^t L(s)\,ds \geqq \int_1^t sl^2(s)\,ds \text{ for } t \geqq 1.$$

By (17) and the case $T = 1$ of (18),

(24) $$\int_1^t sl^2(s)\,ds \geqq (\log mx(t))^2/\log t, \text{ where } m = 1/x(1) > 0.$$

Hence, by (22) and (23),

(25) $$\int_0^t F(s)\,ds \leqq \log x(t) - (\log mx(t))^2/\log t \text{ for } t > 1.$$

Since, by assumption, $x = x(t)$ is of class $L_2(0,\infty)$ and since (5) holds, the product $(x'/x)x = x'$ is of class $L_1(0,\infty)$. Hence $\lim x(t)$, as $t \to \infty$, exists and is 0. In particular, $\log mx(t) < 0$ for large t. Consequently, (24) implies that, for large t,

$$\log mx(t) \geqq -(\log t)^{\frac{1}{2}}\Big(\int_1^t sl^2(s)\,ds\Big)^{\frac{1}{2}},$$

and so

$$\infty > m^2\int^\infty x^2(t)\,dt \geqq \int^\infty \exp\{-2(\log t)^{\frac{1}{2}}\Big(\int_1^t sl^2(s)\,ds\Big)^{\frac{1}{2}}\}dt.$$

The integrand of the last (convergent) integral is monotone decreasing and therefore is $o(1/t)$, as $t \to \infty$ (Abel). Thus, as $t \to \infty$,

$$-2(\log t)^{\frac{1}{2}}\Big(\int_1^t sl^2(s)\,ds\Big)^{\frac{1}{2}} + \log t \to -\infty\,;$$

in particular $\int_1^t sl^2(s)\,ds \geqq \frac{1}{4}\log t$ for sufficiently large t. Hence, by (22) and (23),

$$\int_0^t F(s)\,ds \leqq \log x(t) - \tfrac{1}{4}\log t \text{ for large } t. \tag{26}$$

Let $t_0(>1)$ be chosen so large that the inequalities (25), (26) hold for $t \geqq t_0$. Let the half-line $t_0 \leqq t < \infty$ be divided into two mutually exclusive sets, R and S, by the requirements that

$$mx(t) \leqq t^{-\frac{1}{2}} \text{ if } t \text{ is in } R \tag{27}$$

and

$$mx(t) > t^{-\frac{1}{2}} \text{ if } t \text{ is in } S. \tag{28}$$

In the first case, when (27) holds, $\log mx(t) \leqq -\frac{1}{2}\log t\,(<0)$, which implies that $(\log mx(t))^2/\log t \geqq -\frac{1}{2}\log mx(t)$. Thus, by (25),

$$\int_0^t F(s)\,ds \leqq \log x(t) + \tfrac{1}{2}\log mx(t) = \tfrac{3}{2}\log x(t) + \tfrac{1}{2}\log m$$

if t is in R. Consequently, an integration over the set R gives

$$\int_R \exp\Big(\tfrac{4}{3}\int_0^t F(s)\,ds\Big)\,dt \leqq m^{\frac{4}{3}}\int_R x^2(t)\,dt < \infty. \tag{29}$$

On the other hand, (26) gives

$$\int_S \exp\Big(\tfrac{4}{3}\int_0^t F(s)\,ds\Big)\,dt \leqq \int_S x^{\frac{4}{3}}(t)\,t^{-\frac{1}{3}}dt,$$

while the inequality of Hölder shows that

$$\Big(\int_S x^{\frac{4}{3}}(t)\,t^{-\frac{1}{3}}dt\Big)^3 \leqq \Big(\int_S x^2(t)\,dt\Big)^2\Big(\int_S t^{-1}dt\Big).$$

Finally, (28) implies that $\infty > m^2\int_S x^2dt \geqq \int_S t^{-1}dt$. Hence, by the last

two formula lines,

$$\int_S \exp(\tfrac{4}{3}\int_0^t F(s)\,ds)\,dt < \infty. \tag{30}$$

Since $R + S = [t_0, \infty)$, the assumption, that (12) holds when $B = \frac{4}{3}$, is contradicted by (29) and (30). This completes the proof of the first assertion in (III).

Since the second assertion is a consequence of the examples (2), the proof of (III) is complete.

Remark. The theorem (III) remains true if the number $\frac{4}{3}$ and the formula (11), in the statement of (III), are replaced by the number $(\gamma + \gamma^2)^{-1}$, where $\frac{1}{2} \leqq \gamma \leqq 1$, and the formula (11 bis), respectively.

In view of the examples (2), it is sufficient to consider the first assertion of the modified theorem (III). The derivation of (25) remains valid. A slight modification of the derivation of (26) leads to

$$\int_0^t F(s)\,ds \leqq \log x(t) - \gamma^2 \log t \text{ for large } t. \tag{26 bis}$$

If $t^{-\frac{1}{2}}$, in (27) and (28), is replaced by $t^{-\gamma}$, the modified version of (III) follows from (25) and (26 bis), as (III) does from (25) and (26).

PARIS, FRANCE.

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ERRATA.

P. Hartman, "The number of L^2-solutions of $x'' = q(t)x = 0$," this Journal, vol. 73, pp. 635-645.

Page 637, in the last integral of formula (10), read $dQ^{-2}(s)$ instead of $dQ^2(s)$.

I. I. Hirschman, Jr., "On the distribution of the zeros of functions belonging to certain quasi-analytic classes," this Journal, vol. 72, pp. 396-406.

On page 403, line 3 from bottom, the upper integration limit should read $\mathfrak{H}^*(t)$, instead of u.

On page 404, line 1, omit the factor $\pi/2$.

Aurel Wintner, "On the Hölder restrictions in the theory of partial differential equations," this Journal, vol. 72, pp. 731-738.

Page 733, formula (2), read $\Delta u + \{$ instead of $u + \{$.

Page 733, line 3 from bottom, read (b) instead of (4).

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